International Journal for Research in Engineering Application & Management (IJREAM)
ISSN : 2454-9150 Vol-04, Issue-03, June 2018

On the Additive and Multiplicative structures of Regularity and Unit element

[1]M. Amala, [2]N. Sulochana

[1] Dept. of Mathematics, Sri Padmavathi Mahila Visvavidyalayam, Tirupati, Andhra Pradesh, India

[2] Dept of Mathematics, K.S.R.M College of Engineering, Kadapa, Andhra Pradesh, India.

[1]amalamaduri@gmail.com, [2]sulochananagam@gmail.com

ABSTRACT: In this paper, In first section we have shown that Let S be a semiring and 'a' be additively completely regular element. If (S, •) is a band, then S contains additively periodic elements. In last section It was proved that (S, +) is a band under the following two cases.

(i) If S is a Unit Semiring and multiplicatively subidempotent semiring,

(ii) If 'a' in S is a Unit element and additively idempotent element in a semiring S.

AMS Mathematics Subject Classification (2010): 20M10, 16Y60.

Index terms: left permutable, maximum, positively totally ordered, zero square.

I. INTRODUCTION

There are several concepts of collective algebras to generalize that of a ring (R, +, •). Those are known as semirings, which begin from rings, generally speaking, by cancelling the assumption that (R, +) has to be a group. The semiring theory is attracting the concentration of a number of algebraists due to its applications to Formal language theory, Optimization theory, Computer Science, Automata theory and cryptography. Especially semirings with different constrains have become useful in theoretical Computer Science.

The first formal definition of semiring was introduced in the year 1934 by Vandiver. However the developments of the theory in semirings have been taking place since 1950. Semirings flourish in the mathematical world around us. A semiring is basic structure in Mathematics. Certainly the first mathematical structure we know–the natural number set N is a semiring. Other semirings take place naturally in different areas of mathematics as graph theory, Euclidean geometry, combinatorics, topology, probability theory and the mathematical modeling of quantum physics and parallel computation systems.

Semiring theory stands with a foot in each of two mathematical domains. On one hand, semirings are abstract mathematical structures and their study is a part of abstract algebra arising from the work of Dedekind, Macaulay, Krull and others. On the theory of ideals of a commutative ring and then through the more general work of Vandiver and the tools used to study them is primarily the tools of abstract algebra. In a survey it is observed that the theory of semirings and ordered semirings find wide applications in linear and combinatorial optimization problems such as path problems, transportation and Eigen value problems.

II. PRELIMINARIES

Definition 2.1: A semiring S is zero-square if $a^2 = 0$, for all a in S. Here zero is multiplicative zero.

Definition 2.2: Semigroups (S, +) and (S, •) are bands if a + a = a for all a in S and $a^2 = a$ for all a in S.

Definition 2.3: A semiring S is multiplicatively subidempotent $a + a^2 = a$ for all a in S. **Definition 2.4.:** An element a in a semigroup (S, +) is periodic if ma = na where m and n are positive integers. A semigroup (S, +) is periodic if every one of its elements is periodic.

Definition 2.5: A semigroup (S, •) is left (right) permutable if for every a, x, c in S, axc = acx (axc = xac).

Definition 2.6: A semigroup (S, •) is quasi commutative if $ab = b^n a$ for some integer $n \geq 1$. **Definition 2.7:** An element 'a' in a partially ordered semigroup (S, •, ≤) is non-negative (non-positive) if $a^2 \geq a$ ($a^2 \leq a$). A partially ordered semigroup (S, •, ≤) is non-negatively ordered (non-positively ordered) if every element of S is non-negative (non-positive).

Definition 2.8: An element x in a t.o.s.r is minimum (maximum) if $x \leq a$ ($x \geq a$) for all a in S.

III. COMPLETELY REGULAR SEMIGROUP

It is well known that, regular semigroup plays a prominent role in examining semigroups. One area of research in the field of semigroup theory in which there have been significant success in recent years has been the subject of completely regular semigroups. In the very last decades, generalizations of this course group of semigroups are extensively investigated by several authors and plenty of exciting results have been acquired. The class of

 DOI : 10.18231/2454-9150.2018.0406

C.R.Semigroups is one of the most studied classes of semigroups because they permit different structural descriptions. The first paper on this C.R.Semigroup was published by Clifford. In his paper the first decompositions into groups and completely simple semigroups are given which justified its alternative name as "union of groups".

In a semigroup (S, +) an element 'a' is said to be additively completely regular (C.R) element if there exists an element x such that a + x + a = a and a + x = x + a.

Theorem 3.1: Suppose S is a zero square semiring and 'a' is additively completely regular element. Then ax = xa = 0 where 'x' depends on 'a'.

Proof: By hypothesis 'a' is additively completely regular element then there exists 'x' such that a + x + a = a → **(1)**

Using zero square semiring in above equation we get ax = 0

Therefore ax = 0 and similarly we can prove that xa = 0

Proposition 3.2: Let S be a semiring and 'a' be additively completely regular element. If (S, •) is a band, then S contains additively periodic elements.

Proof: From hypothesis 'a' is additively completely regular element, thus we obtain $a^2 + ax + a^2 = a^2$

Since (S, •) is a band then a + ax + a = a implies ax + ax + ax = ax ⇒ 3ax = ax

Thus an element 'ax' is periodic

Again by considering a + x + a = a we obtain $a^2x + xax + a^2x = a^2x$

⇒ ax + xax + ax = ax ⇒ $xax + x^2ax + xax = xax$ ⇒ 3xax = xax

Thus an element 'xax' is periodic

Similarly we can prove that 3axa = axa thus an element 'axa' is periodic

Hence S contains additively periodic elements

Theorem 3.3: If S is a semiring and 'a' is additively completely regular element, then na + x = (n -1) a and a + n (x + a) = a where 'x' depends on 'a' and n >1.

Proof: Given that 'a' is additively completely regular element then there exists 'x' such that a + x + a = a and a + x = x + a

From above equations we obtain a + a + x = a

Again by performing some addition and multiplications operations on above equation we get 3a + x = 2a

Proceeding in a similar manner we get na + x = (n -1) a for n >1

Again by considering a + x + a = a we obtain a + 3 (x + a) = a

Continuing like this we get a + n (x + a) = a for n >1

Proposition 3.4: Let S be a totally ordered semiring. and 'a' be an additively completely regular element. If (S, +) is positively totally ordered, then a = a + a = a + x where 'x' depends on 'a'.

Proof: By hypothesis 'a' is additively completely regular element then there exists an element 'x' such that a + x + a = a and a + x = x + a → **(1)**

Given that (S, +) is positively totally ordered then a + x ≥ a and a + x ≥ x

⇒ a + x + a ≥ a + a and a + x + a ≥ x + a → **(2)**

From equations (1) and (2) we get a ≥ a + a and a ≥ a + x → **(3)**

Since (S, +) is p.t.o a + a ≥ a and a + x ≥ a → **(4)**

Comparing equations (3) and (4) a ≥ a + a ≥ a and a ≥ a + x ≥ a

Therefore a = a + x = a + a

IV. UNIT SEMIRING: INTRODUCTION

An element *a* of a semiring is "unit" if and only if there exists an element x satisfying the condition ax = xa =1.The element a is called the inverse of x. If such an inverse exists for a unit, it must be unique. We will normally denote the inverse of element a by an element x. The set of all units of S are denoted by U(S). This set is non-empty, since it contains "1" & is not all of S, since it does not contain '0'.

In a semiring S an element **a** is **Unit** if there exists an element x in S such that ax = xa =1. Here '1' is multiplicative identity.

Proposition 4.1: Let 'a' be unit element in a semiring S and (S, •) be left permutable. Then $a^ncx^n = c$ for all c in S and n ≥ 1, where 'x' depends on 'a'.

Proof: By hypothesis 'a' is unit element then there exists 'x' such that ax = xa = 1

Since (S, •) is left permutable axc = acx for every a, x, c in S

⇒ axc = c ∀ c ∈ S implies acx = c

⇒ $a^4cx^4 = a^3cx^3 = a^2cx^2 = acx = c$

By generalizing the above we get $a^ncx^n = c$ ∀ c ∈ S and for n ≥1

Hence $a^ncx^n = c$ for all c in S and n ≥ 1

Theorem 4.2: Suppose S is a Unit semiring

(i) If (S, •) is quasi commutative, then (S, •) is periodic.

(ii) If (S, +) is cancellative, then (S, +) is commutative.

Proof: **(i)** Assume that S is a unit semiring then ax = xa = 1

By the definition of (S, •) quasi commutative $ax = x^m a$ for n > 1.

This implies $x^m a = 1 = xa$ ⇒ $x^m = x$ for all x in S

Thus x is a periodic element Therefore (S, •) is periodic

(ii) First let us consider the term x + a + x + a

 DOI : 10.18231/2454-9150.2018.0406

Using multiplicative identity 1 in above we obtain x.1 + a.1 + x.1 + a.1

Which is equal to (x + a) (1 + 1)

The above equation can also be written as x (1 + 1) + a (1 + 1)

Thus x + a + x + a = x + x + a + a

By using cancellation property we get a + x = x + a

Therefore (S, +) is commutative

Theorem 4.3: (S, +) is a band under the following cases.

(i) If S is a Unit Semiring and multiplicatively subidempotent semiring,

(ii) If 'a' in S is a Unit element and additively idempotent element in a semiring S.

Proof: (i) Given that S is unit semiring then ax = xa =1 → **(1)**

Also S is multiplicatively subidempotent then $a + a^2 = a$ and $x + x^2 = x$

$\Rightarrow (a + a^2)(x + x^2) = ax \Rightarrow ax + a^2x + ax^2 + a^2x^2 = ax$ → **(2)**

From above two equations we obtain 1 + a + x + 1 = 1 → **(3)**

Since $a + a^2 = a$ implies $ax + a^2x = ax \Rightarrow 1 + a = 1$ for all a in S → **(4)**

From equations (3) and (4) we get 1 + x + 1 = 1 ⇒ 1 + 1 = 1

Thus b + b = b for all b in S Hence (S, +) is a band

(ii) Since 'a' is a unit element then there exists x in S such that ax = xa = 1

Given that 'a' in S is an additive idempotent element then a + a = a

⇒ ax + ax = ax ⇒ 1 + 1 = 1 This implies b + b = b for all b in S

Therefore (S, +) is a band

Proposition 4.4: Let S be a totally ordered semiring,

(i) If 'a' is Unit and (S, +) is positively ordered, then a + 1 = 1.

(ii) If (S, •) is non-negatively ordered (non-positively ordered), then 1 is the minimum (maximum) element.

Proof: (i) By hypothesis (S, +) is positively ordered, a + x ≥ a, x for all a, x in S

⇒ a + x + ax ≥ a + ax implies ax ≥ a + ax ⇒ 1 ≥ a + 1 → **(1)**

Also (S, +) is positively ordered a + 1 ≥1 → **(2)**

From equations (1) and (2) a + 1 = 1

(ii) By hypothesis (S, •) is non-negatively ordered then $a^2 \geq a$ for all a in S

This implies $a^2x \geq 1 \Rightarrow a(ax) \geq 1$

Since S is unit semiring then ax = xa = 1 then above equation becomes a.1 ≥ 1 ⇒ 1 ≤ a for all a in S Thus 1 is the minimum element

Similarly 1 is the maximum element if (S, •) is non-negatively ordered

REFERENCES

[1] V.R.Daddi and Y.S.Pawar, "On Completely Regular Ternary Semiring", Novi Sad J.Math. ol. 42, No. 2, 2012, 1-7.

[2] Jonathan S.Golan, "Semirings and Affine Equations over Them: Theory and Applications", Kluwer Academic publishers.

[3] Jonathan S.Golan, "Semirings and their Applications" , Kluwer Academic Publishers.

[4] M.P.Grillet , "Semirings with a completely semisimple additive semigroups". Jour. Aust. Math. Soc., Vol.20 (1975), 257-267.

[5] Kanchan Jana, "Quasi K-Ideals In K-Regular And Intra K-Regular Semirings", Pu. M.A.Vol. 22 (2011), No.1, Pp. 65-74.

[6] N.Kehayopulu, "On a characterization of regular duo le Semi groups", Maths. Balkonica (1977), 181- 186.

[7] M.Satyanarayana, "Naturally totally ordered semigroups" Pacific Journal of Mathematics, Vol.77, No. 1, 1978.

[8] M.Satyanarayana, "Positively ordered semigroups". Lecture notes in Pure and Applied Mathematics, Marcel Dekker, Inc., Vol.42 (1979).

[9] M.K.Sen, S.K.Maity and K.P. Shum, "Clifford Semirings And Generalized Clifford Semirings" Taiwanese Journal of Mathematics Vol. 9, No. 3, pp. 433-444, September 2005.

[10] M.K.Sen, S.K.Maity, "Semirings Embedded in a Completely Regular Semiring", Acta Univ. Palacki. Olomuc., Fac.rer. nat., Mathematica 43 (2004) 141–146.

[11] K.P.Shum, "Topological semi primary semigroups", Studia, Sci.Math., Hungar 15 (1980), No.1-3, 15-18.

[12] K.P.Shum and C.S.Hoo, "Prime radical theorem on ordered Semigroup", Semigroup Forum No.19 (1980), No.1, 87-94.

[13] K.P.Shum and C.Y.Hung, "Topological radicals in semigroups", Publ. Math. Debrecen, 29 (1982), No.3-4, 265-274.

[14] H.E.Stone, "Ideals in half rings", Proc. Amer. Math.Soc. Vol.33 (1972), 8-14.

[15] G.Theirrin, "The Sytantic monoid of a hypercode", Semigroup Forum 6 (1973), 227 - 231.

 DOI : 10.18231/2454-9150.2018.0406