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Refinements and Generalizations of Gauss Lucas Theorem

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Abstract The classical Gauss-Lucas theorem states that the roots of derivative of non constant polynomial $F(z)$ lie in the convex hull of zeros of $F(z)$. This theorem has been widely used to study different properties of polynomials. The present paper provides the various refinements and generalizations of Gauss Lucas Theorem.

Keywords- Gauss-Lucas theorem, Polynomial, Convex hull, Zeros.

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I. INTRODUCTION

Gauss-Lucas theorem is named after Carl Friedrich Gauss and Felix Lucas. Gauss-Lucas theorem establish a relation between the roots of a polynomial F and the roots of F' i.e. the roots of derivatives of F. According to Gauss-Lucas theorem, the roots of derivatives of F lie in convex hull of roots of F.

Definition 1.1 (Convex hull): Convex hull C(S) of any convex set S is the smallest convex polygon that contains S. In other words, convex hull is the intersection of all convex sets containing S. Convex hull of a set S may also be defined as the set of all convex combinations of finite points of S. So,

$$C(S) = \{ \textstyle\sum_{i=1}^{n} x_i\, a_i \ ;\ a_i \geq 0, \sum_{i=1}^{n} a_i = 1 \ , x_i \text{ is in S}\} \qquad (1.1)$$

If S contains only one point then convex hull of S is a single point. If the points of S lie on a line then its convex hull is the line segment formed by joining two extreme points of S. If the points of S are some points of the plane then convex hull of S can be thought of a shape of a rubber band enclosing all the points of S and stretched around S. Gauss-Lucas theorem has very important physical interpretation in potential theory. This theorem can be considered as a variant of Rolle's theorem for complex polynomial[1]. This result has been widely used to study the properties of the polynomials[2].

Theorem 1.1 (Gauss-Lucas Theorem): All the critical points of $F(z)$ i.e. the zeros of the derivatives $F'(z)$ lie in the convex hull C(S) of the zeros of F(z) where $F(z)$ is non-constant univariate polynomial and S = $\{z_1, z_2, \ldots\ldots, z_n\}$ is the set of zeros of F(z). Further if the zeros of F(z) are non-collinear then no critical point of $F(z)$ can lie on the boundary of C(S) unless it is a multiple root of $F(z)$.

This result was given by Gauss and later explicitly stated and proved by Lucas[3] in 1879. In 1836, Gauss proved that the roots of $F'(z)$ which are different from multiple roots of $F(z)$ can be considered as the points of equilibrium for the fields of forces created by unit masses placed at the roots of $F(z)$ which repel or attract with a force inversely proportional to the distance of this unit mass.

II. SOME REFINEMENTS OF GAUSS-LUCAS THEOREM

2.1 Refinement of Gauss-Lucas theorem (Dimitrov[4])

The convex hull C(S) as mentioned in Gauss-Lucas theorem contains all the zeros of $F_n{}'(z)$. In fact a sub domain of C(S) contains all the zeros of $F_n{}'(z)$ as described by D.K. Dimitrov[4]. Let $F_n(z)$ be a polynomial with roots $z_1, z_2, \ldots.. z_k$ with multiplicities $m_1, m_2, \ldots.. m_k$ respectively.

$$F_n(z) = \prod_{i=1}^{k} (z - z_i)^{m_i}, \quad \sum_{i=1}^{k} m_i = n \qquad (2.1.1)$$

Corresponding to every zero of $F_n(z)$, z_i with multiplicity m_i, we associate a closed circular region C_i. This region C_i in complex plane is such that it contains the points 1 and $\frac{m_i}{n}$. For each $r \neq i$, T_{ri} is the affine transformation of C_i given by

$$T_{ir} = z_i + (z_r - z_i)\, c_i \qquad (2.1.2)$$

And define

$$T_r = \bigcup_{r \neq i} T_{ir} \qquad (2.1.3)$$

This refinement of Gauss-Lucas theorem as given by Dimitrov[4] is as follows:

Theorem 2.1.1 (Dimitrov[4]): Every zero of $F_n{}'(z)$, not coincident with z_i lies in T_r . Further if T_r: r =1,2,3,….,k are all regions associated with distinct roots of $F_n(z)$ then

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every non trivial zero of $F_n'(z)$ lies in the region

$$T(\mathrm{F(z)}) = \bigcap_{r=1}^{k} T_r.$$

2.2 Refinement of Gauss-Lucas theorem(Arnaud et. al.[5])

W. P. Thurston gave a geometric proof of classical Gauss-Lucas theorem using the concept of distance function and considering the polynomial as product of linear factors. He gave another version of the classical Gauss-Lucas theorem as follows:

Theorem 2.2.1 (A surjective version of classical Gauss-Lucas theorem[5]): Let $F_n(z)$ be a polynomial of degree n where n is atleast 2 and C(S) be the convex hull of zeros of $F_n'(z)$ where S = {z_1, z_2,……., z_n} is the set containing zeros of $F_n'(z)$. Then F_n : E →C is a surjective mapping for any closed half plane E intersecting C(S).

Theorem 2.2.2 (Gauss-Lucas-Thurston[5]): Let $F_n(z)$ be a polynomial of degree n where n is atleast 2 and C(S) be the convex hull of zeros of $F_n'(z)$ where S = {z_1, z_2,……., z_n} is the set containing zeros of $F_n'(z)$. Let L be a straight line intersecting C(S) and let it bounds an open half plane H which is disjoint from C(S). If a point a on L is zero of $F_n'(z)$ then there exist a set of geodesic for metric of $|F_n'(z)|\,|dz|$ originating from a on L and in the direction of H so that it forms an open subset X of H for which F($\bar{X}$)= C.

III. SOME EXTENSIONS OF GAUSS-LUCAS THEOREM

3.1 Extension of the Gauss-Lucas theorem to convex linear combinations of incomplete polynomials (J.L. Diaz-Barrero and J.J. Egozcue[6])

This generalization of Gauss-Lucas theorem given by J.L. Diaz-Barrero and J.J. Egozcue[6] is the extension of classical Gauss-Lucas theorem to convex linear combinations of incomplete polynomials.

Definition 3.1.1 (Incomplete Polynomials): An incomplete polynomial is a polynomial which has some coefficients as zero. The general representation of an incomplete monic polynomial is of the form

$$h_r(\mathrm{z}) = \prod_{\substack{i=1 \\ i\neq r}}^{n} (\mathrm{z}-\mathrm{z}_i) \tag{3.1.1}$$

Here r is any integer between 0 and n corresponding to which the linear factor (z – z $_r$) in the continuous product of factors (z – z_i); (i = 1 to n) is missing. Clearly h_r(z) is of degree n-1. If $F_n(z)$ is a monic polynomial with roots z_1,z_2,….,z_n then $F_n(z)$ is of the form

$$F_n(\mathrm{z}) = \prod_{i=1}^{n} (\mathrm{z}-\mathrm{z}_i) \tag{3.1.2}$$

If the derivative of $F_n(z)$ is normalized to form a monic polynomial then

$$\frac{1}{n} F_n'(\mathrm{z}) = \sum_{j=1}^{n} \frac{1}{n} h_j(\mathrm{z}) \tag{3.1.3}$$

Clearly the R.H.S. of (3.1.3) indicates that $\frac{1}{n}F_n'(z)$ is a convex linear combination of incomplete polynomials as each $\frac{1}{n}$ lies between 0 and 1 and $\frac{1}{n}+\frac{1}{n}+\frac{1}{n}+\cdots n\ times=1$. Obviously the derivative of $F_n(z)$when reduced to monic polynomial is one of the convex linear combinations of incomplete polynomial. This fact leads to the following theorem which is an extension of the Gauss-Lucas theorem to convex linear combinations of incomplete polynomials.

Theorem 3.1.1 (J.L. Diaz-Barrero and J.J. Egozcue[6]): If $F_n(z)$ is a monic polynomial with complex coefficients with zeros $\mathrm{z}_1,\mathrm{z}_2,\ldots,\mathrm{z}_k$ with multiplicities $m_1, m_2, \ldots, m_k$ such that $\sum_{i=1}^{n} m_i = n$ and let $F_n(\mathrm{z})$ be a polynomial of degree $n-1$ such that

$$\oint_{\tau} \frac{F_{n-1}(\mathrm{z})}{F_n(\mathrm{z})} dz = 2\pi i \tag{3.1.4}$$

And

$$\operatorname*{Res}_{z=z_i} \left\{ \frac{F_{n-1}(\mathrm{z})}{F_n(\mathrm{z})} \right\} \geq 0,\ \ i = 1, 2, \ldots, k \tag{3.1.5}$$

Then all the zeros of $F_{n-1}(z)$ must lie in the convex hull C(S) where S = {z_1, z_2,……., z_n} and τ is the closed path containing all the zeros of $F_n(z)$.

3.2 Extension of Gauss-Lucas theorem for integral functions(Moreno[7])

Definition 3.2.1 (Integral function): An integral function is a complex valued function that is analytic at all the points over the whole complex plane. Integral function is also called entire function[8].

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Definition 3.2.2 (Hurwitz Stable Polynomial): A Hurwitz Stable Polynomial is a polynomial whose zeros lie either in left half plane of whole complex plane or on imaginary axis. In other words, the real part of every zero is either zero or negative [9].This polynomial is named after Adolf Hurwitz (1859-1919), a German Mathematician.

Definition 3.2.3 (Derivative of Hurwitz Stable Polynomial[7]): The derivative of Hurwitz Stable Polynomial $F(z)=\sum_{i=1}^{n} a_i z^i$ is given by

$$F'(z)=\sum_{i=0}^{n-1}(i+1)a_{i+1}z^i$$

which is also a polynomial of lower degree.

An important property of stable polynomial which is a consequence of Gauss Lucas Theorem is that if F(z) is stable then F'(z) is also stable. The validity of this property is demonstrated by J. Moreno [7].

Definition 3.2.4 (Quasi polynomial): A quasi polynomial is a generalization of polynomial in which the coefficients are periodic functions with integral period.

Theorem 3.2.1 (Hermite Biehler theorem): If a polynomial F(z) is stable then it satisfies monotone phase increasing property i.e.the argument of F(ωi) is monotone increasing function.

Definition 3.2.5 (UMPLIP): An entire function is said to satisfy uniform monotone phase increasing property(UMPIP) if the argument of F($\rho + \omega i$) is monotone increasing function for every $\rho \geq 0$.

Theorem 3.2.2 (Moreno[7]): If the entire function F(z) is stable and satisfies UMPIP then F'(z) is also stable.

Moreno[7] extended the implication of Lucas theorem (i.e. derivative of Hurwitz stable polynomial is also Hurwitz stable polynomial) to the entire functions provided that it satisfy uniform monotone phase increasing property(UMPIP).

3.3 Extension of Gauss-Lucas theorem from the set of complex numbers to the set of bicomplex numbers (M. Bidkham and S. Ahmadi [10])

This section provides the extension of the Gauss-Lucas theorem from the set of complex numbers to the set of bicomplex numbers as given by M. Bidkham and S. Ahmadi[10].

Definition 3.3.1 (Bicomplex numbers): A bicomplex number is the number of the form $a_1+ib_1+j(a_2+ib_2)$ where $a_1,b_1,a_2,b_2\in R$, $i^2=-1$, $j^2=-1$ and $ij=ji$.

We can also write a bicomplex number as z_1+jz_2 where z_1 and z_2 are complex numbers.

Following are some important properties of bicomplex numbers:

1. A bicomplex number z_1+jz_2 can be uniquely expressed as
$$(z_1-iz_2)e_1+(z_1+iz_2)e_2=\lambda e_1+\mu e_2$$
Where $e_1=\frac{1+ij}{2}$, $e_2=\frac{1-ij}{2}$
2. Let $z=\lambda e_1+\mu e_2$, $z'=\lambda' e_1+\mu' e_2$
Then
$z+z'=(\lambda+\lambda')e_1+(\mu+\mu')e_2$,
$zz'=(\lambda\lambda')e_1+(\mu\mu')e_2$
3. If $z=\lambda e_1+\mu e_2$ then z is invertible if z^{-1} exists
So
$zz^{-1}=1$
$and\ z^{-1}=\frac{1}{\lambda}e_1+\frac{1}{\mu}e_2$
Provided that $\lambda,\mu\neq 0$
4. The usual norm of z is given by
$$|z|=|\lambda e_1+\mu e_2|=\sqrt{\frac{|\lambda|^2+|\mu|^2}{2}}$$

Definition 3.3.2 (Cartesian set): The Cartesian set in the set of bicomplex numbers determined by X_1 and X_2 is defined by

$$X=X_1\times_e X_2$$
$$=\{z_1+jz_2 : z_1+jz_2=\lambda e_1+\mu e_2, (\lambda,\mu)\in X_1\times X_2\}$$

Definition 3.3.3 (disk) Let $z'=\lambda' e_1+\mu' e_2$ be a fixed bicomplex number. An open disk with center z' and radii r_1 and r_2 is given by

$$D(z':r_1,r_2)$$
$$=\{z_1+jz_2 : z_1,z_2\in C, z_1+jz_2=\lambda e_1+\mu e_2, |\lambda-\lambda'|<r_1, |\mu-\mu'|<r_1\}$$

and a closed disk with center z' and radii r_1 and r_2 is given by

$$\overline{D}(z':r_1,r_2)=$$
$$\{z_1+jz_2 : z_1,z_2\in C, z_1+jz_2=\lambda e_1+\mu e_2, |\lambda-\lambda'|\leq r_1, |\mu-\mu'|\leq r_1\}$$

The extension of Gauss-Lucas theorem for bicomplex

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numbers is given by M. Bidkham and S. Ahmadi[10] is based on following lemmas:

Lemma 3.3.1 : The Cartesian product in the set of bicomplex numbers determined by X_1 and X_2 is convex set in C.

Lemma3.3.2 : If $X_1 = \{a_1, a_2, ...a_n; a_i \in C\}$ and $X_2 = \{b_1, b_2, ...b_m; b_i \in C\}$ then

(i) $C(X_1 \times_e X_2) = C(X_1) \times_e C(X_2)$

(ii) $C(X_1 \times_e C) = C(X_1) \times_e C$

(iii) $C(C \times_e X_2) = C \times_e C(X_2)$

Lemma 3.3.3 : In a bicomplex polynomial of the form $F(z) = \sum_{i=1}^{n} a_i z^i$ if all the coefficients a_i are multiples of e_1 and the constant term $a_0 = \lambda_0 e_1 + \mu_0 e_2$ has $\lambda_0 \neq 0$ then F(z) has no root otherwise F(z) has atleast one root.

This lemma is analogue of fundamental theorem of algebra for bicomplex theorem. Based on above lemmas, following is the extension of Gauss-Lucas theorem:

Theorem 3.3.1 (Bidkham and Ahmadi[10]): If F(z) is a non constant bicomplex polynomial with atleast one zero then every zero of F'(z) lies in the convex hull of zeros of F(z).

3.4 Generalization of Gauss-Lucas theorem on a circle(A.Aziz and B.A.Zargar [11])

This generalization is given by A.Aziz and B.A.Zargar[11] in the following theorem:

Theorem 3.4.1 (A.Aziz and B.A.Zargar [11]): If all the zeros of a polynomial F(z) lie in a disk $|z-a| \leq R$ and ω is any number real or complex, satisfying

$$|(\omega - a)F'(\omega)| \leq |(\omega - a)F'(\omega) - nF(\omega)| \quad (3.4.1)$$

Then $|\omega - a| \leq R$

(3.4.2)

where n is the degree of F(z)

If (3.4.1) is satisfied as strict inequality then (3.4.2) is also satisfied as strict inequality. Gauss Lucas theorem can be considered as a special case of above theorem. If we consider all the zeros of F(z) to lie in the circle $|z-a| \leq R$ and taking ω as a root of F'(z) = 0 then (3.4.1) is obviously true. So we have $|\omega - a| \leq R$ i.e. all the zeros of F'(z) lie in the circle $|z-a| \leq R$.

3.5 Generalization of Gauss-Lucas theorem in asymptotic sense(V. Totik[12]):

The generalization of the classical Gauss-Lucas theorem in asymptotic sense shows that if almost all the zeros of F(z) lie in the convex set then almost all the zeros of F'(z) lie in any fixed neighborhood of that convex set.

Theorem 3.5.1 (V. Totik[12]): If F(z) has almost all its zeros on compact convex set K then for every ε > 0, almost all the zeros of F'(z) lie on K_ε , the ε-neighborhood of K.

The ε-neighborhood of K is considered to be slightly larger than the set K.

In 2003, S.Malamud and R.Pereira gave an extension of Gauss-Lucas theorem using the theory of majorization of sequences.

Theorem 3.5.2 (Malamud-Pereira[13]): If $X = (x_1, x_2, ..., x_n)$ are the zeros of polynomial F(z) of degree n and $Y = (y_1, y_2, ..., y_{n-1})$ are the zeros of F'(z) then there exist a doubly stochastic matrix A of order (n-1) x n such that $y = Ax$.

A rectangular matrix of order (n-1) x n , A=[a_{ij}] is said to be doubly stochastic if $a_{ij} \geq 0$ and $\sum_{j=1}^{n} a_{ij} = 1$ and $\sum_{i=1}^{n-1} a_{ij} = \frac{n-1}{n}$

This Malamud-Pereira theorem implies above theorem provided that all the zeros of F(z) lie in a fixed compact set but Malamud-Pereira theorem does not imply above theorem in full sense.

3.6 Extension of Gauss-Lucas theorem in higher dimension Euclidean space (A.W. Goodman [14])

An extension of Gauss-Lucas theorem in higher dimensional Euclidean space is described by A.W. Goodman [14].

Let $P_j = \left(x_1^{(j)}, x_2^{(j)}, ..., x_n^{(j)}\right), j = 1, 2, ..., m$ be distinct points in *n*-dimensional Euclidean space.

Let the Euclidean distance r_j of P from P_j be defined by

$$r_j = |PP_j| = \left[\sum_{k=1}^{n} \left(x_k - x_k^{(j)}\right)^2\right]^{1/2}$$

Further consider the function

$$G(x) = \sum_{j=1}^{m} g_j(r_j)$$

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where $X=(x_1,x_2,..x_n)$ and $g_j(r_j)$ is the function associated with P_j given by $\log r_j$.

So if $F(z)=\prod_{j=1}^{m}(z-z_j)$, then

$G(X)=\log|F(z)|$ in $E_2: z=x_1+ix_2$

Now

$$\nabla G(X)=\sum_{j=1}^{n}\frac{\partial}{\partial x_1}\log|z-z_j|e_1+\sum_{j=1}^{m}\frac{\partial}{\partial x_2}\log|z-z_j|e_2$$

$$=\sum_{j=1}^{m}\frac{z-z_j}{|z-z_j|^2}$$

$$=\overline{\left(\frac{F'(z)}{F(z)}\right)}$$

For a zero of $F'(z), \nabla G(X)=0.$

Using these notations, the extension of Gauss-Lucas Theorem as given by Dioz and Shaffer [15] is presented in following theorem:

Theorem 3.6.1 (Dioz and Shaffer [15]):Let $P_1,P_2,\ldots,P_m$ be distance points in E_n and with each point P_j, associate a function $g_j(r)$ such that $g_j'(r)>0$ and continuous for *r>0.*

If X is a zero of vector function $\nabla G(X)=\sum_{i=1}^{n}\frac{\partial G}{\partial x_i}e_i$,

then X lies in the convex hull of set of points $P_1,P_2,\ldots,P_m$

IV. CONCLUSION

The Gauss-Lucas Theorem is well known result in classical complex analysis. This paper provides a review of following refinements Gauss Lucas theorem:

(i) Every zero of $F_n'(z)$, not coincident with z_i lies in T_r. Further if T_r: r =1,2,3,….,k are all regions associated with distinct roots of $F_n(z)$ then every non trivial zero of $F_n'(z)$ lies in the region $T(\mathrm{F(z)})=\bigcap_{r=1}^{k}T_r$.

(ii) Let $F_n(z)$ be a polynomial of degree n where n is atleast 2 and C(S) be the convex hull of zeros of $F_n'(z)$ where S = {z_1, z_2,……., z_n} is the set containing zeros of $F_n'(z)$. Let L be a straight line intersecting C(S) and let it bounds an open half plane H which is disjoint from C(S). If a point *a* on L is zero of $F_n'(z)$ then there exist a set of geodesic for metric of $|F_n'(z)|\,|dz|$ originating from *a* on L and in the direction of H so that it forms an open subset X of H for which F($\bar{X}$)= C.

Further this paper also provides a review of extension of Gauss Lucas theorem on incomplete polynomials, extension of Gauss Lucas theorem on integral functions or entire functions *w.r.t.* hurwitz stable polynomials, extension of Gauss Lucas theorem from the set of complex numbers to the set of bicomplex numbers, generalization of Gauss-Lucas theorem on a circle, generalization of Gauss-Lucas theorem in asymptotic sense and extension of Gauss-Lucas theorem in higher dimension Euclidean space. Despite its simplicity, it is very useful and versatile theorem in complex analysis.

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