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On Permutation Groups and Fourier Matrices

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Abstract: In this paper Fourier matrices of prime order 3, 5 and 7 are constructed with the help of suitable combination of matrix representation of some suitable permutation groups.

Keywords —Fourier Matrix, Hadamard Matrix, Complex Hadamard Matrix, Permutation Group.

I. INTRODUCTION

Fourier matrices are complex Hadamard matrices and they have many applications in conformal field theory. Conformal field theory has many applications in Physics.

(vide [1], [2], [4], [7] and [8])

An square matrix of size n with entries 1 and -1 is called Hadamard matrix if $HH^T = nI_n$.

Following are some examples of Hadamard matrices:

$$[1], \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

The order of Hadamard matrix is of the form 4t, where t is a positive integer. But whether there exist a Hadamard matrix of every order 4t? It is one of the longest-standing open problem in Mathematics.

(vide [5])

A complex Hadamard matrix M is a square matrix of order n with unimodular entries such that $MM^{\ominus}$ where $M^{\ominus}$ is transconjugate of matrix M.

Following are the examples of some complex Hadamard matrices:

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

A complex Hadamard matrix is called dephased if all the entries in its first row and first column are 1 and it is known that every Hadamard matrix is equivalent to a dephased.

Fourier matrices are particular Hadamard matrices defined as a square matrix F_n of size n and of the form

$$F_n = \begin{bmatrix} 1 & e^T \\ e & Q \end{bmatrix}$$

Where e is an (n-1)×1 column matrix with each entry 1, e^T is transpose of matrix e and Q is an (n-1)×(n-1) square matrix defined as

$$Q = (q_{ij}) \ \ and \ \ q_{ij} = e^{\frac{2\pi ij}{n}}, \ i, j \in \{1,2,3,....,(n-1)\}$$

Following are the some examples of Fourier matrices:

$$F_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}, \ F_5 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 \\ 1 & \omega^2 & \omega^4 & \omega^2 & \omega \\ 1 & \omega^3 & \omega & \omega^4 & \omega^2 \\ 1 & \omega^4 & \omega^3 & \omega^2 & \omega \end{bmatrix}$$

(Vide [3], [5],[9], [10] and [11])

II. MAIN WORK

Manjhi and Kumar [6] introduce some methods of constructions of Fourier matrices F_3, F_5 and F_7 with the help of Coherent configuration by taking suitable combinations of adjacency matrices of these Coherent Configurations.

I this paper methods of construction of Fourier matrices F_3, F_5 and F_7 are introduced by the use of suitable permutation group. In the forwarded methods the methods of construction of core matrix Q is given.

1. Construction of Fourier matrix F_3

Consider the permutation group G={I, (12)} over the set of symbols {1, 2}.

The matrix representation of representation of elements of G are given below:

$$I \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = M_1(Say)$$

$$(12) = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = M_2(Say)$$

Now consider the linear combination

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$$\sum_{i=1}^{2}\omega^i M_i = \omega M_1 + \omega^2 M_2$$

$$= \omega\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \omega^2\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

$$= \begin{bmatrix}\omega & \omega^2\\ \omega^2 & \omega\end{bmatrix}$$

$$= Q \quad \text{of} \quad F_3$$

2. Construction of Fourier Matrices F_5 of order 5

Consider the cyclic permutation group

G= {I, (1342), (1243), (14) (23)} over the set of symbols X={1, 2, 3, 4}.

The matrix representation of representation of elements of G are given below:

$$I = \begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 2 & 3 & 4\end{pmatrix} \rightarrow \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} = M_1(Say)$$

$$(1342) = \begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 1 & 4 & 2\end{pmatrix} \rightarrow \begin{bmatrix}0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 0\end{bmatrix} = M_2(Say)$$

$$(1243) = \begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 4 & 1 & 3\end{pmatrix} \rightarrow \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix} = M_3(Say)$$

$$(14)(23) = \begin{pmatrix}1 & 2 & 3 & 4\\ 4 & 3 & 2 & 1\end{pmatrix} \rightarrow \begin{bmatrix}0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\end{bmatrix} = M_4(Say)$$

Now consider the following combination

$$\sum_{i=1}^{4}\omega^i M_i = \omega M_1 + \omega^2 M_2 + \omega^3 M_3 + \omega^4 M_4$$

$$= \omega\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} + \omega^2\begin{bmatrix}0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 0\end{bmatrix}$$

$$+ \omega^3\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix} + \omega^4\begin{bmatrix}0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\end{bmatrix}$$

$$= \begin{bmatrix}\omega & \omega^3 & \omega^2 & \omega^4\\ \omega^2 & \omega & \omega^4 & \omega^3\\ \omega^3 & \omega^4 & \omega & \omega^2\\ \omega^4 & \omega^2 & \omega^3 & \omega\end{bmatrix} = Q'$$

this matrix is equivalent to Q of Fourier matrix F_5 as by the by the permutations of 2nd and 3rd columns in Q' we get the matrix Q of F_5 .

3. Construction of Fourier matrix F_7

Consider the cyclic permutation group

G={I, (142)(356), (154623), (124)(365), (132645), (16)(25)(34)}=<(132645)> over the set of symbols X={1,2,3,4,6,6}

The matrix representation of representation of elements of G are given below:

$$I = \begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6\\ 1 & 2 & 3 & 4 & 5 & 6\end{pmatrix}$$

$$\rightarrow \begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix} = M_2(Say)$$

$$(142)(356) = \begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6\\ 4 & 1 & 5 & 2 & 6 & 3\end{pmatrix}$$

$$\rightarrow \begin{bmatrix}0 & 0 & 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0 & 0 & 0\end{bmatrix} = M_2(Say)$$

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$$(154623)=\begin{pmatrix}1&2&3&4&5&6\\5&3&1&6&4&2\end{pmatrix}$$

$$\to\begin{bmatrix}0&0&0&0&1&0\\0&0&1&0&0&0\\1&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&1&0&0\\0&1&0&0&0&0\end{bmatrix}=M_3(Say)$$

$$(124)(365)=\begin{pmatrix}1&2&3&4&5&6\\2&4&6&1&3&5\end{pmatrix}$$

$$\to\begin{bmatrix}0&1&0&0&0&0\\0&0&0&1&0&0\\0&0&0&0&0&1\\1&0&0&0&0&0\\0&0&1&0&0&0\\0&0&0&0&1&0\end{bmatrix}=M_4(Say)$$

$$(132645)=\begin{pmatrix}1&2&3&4&5&6\\3&6&2&5&1&4\end{pmatrix}$$

$$\to\begin{bmatrix}0&0&1&0&0&0\\0&0&0&0&0&1\\0&1&0&0&0&0\\0&0&0&0&1&0\\1&0&0&0&0&0\\0&0&0&1&0&0\end{bmatrix}=M_5(Say)$$

$$(16)(25)(34)=\begin{pmatrix}1&2&3&4&5&6\\6&5&4&3&2&1\end{pmatrix}$$

$$\to\begin{bmatrix}0&0&0&0&0&1\\0&0&0&0&1&0\\0&0&0&1&0&0\\0&0&1&0&0&0\\0&1&0&0&0&0\\1&0&0&0&0&0\end{bmatrix}=M_6(Say)$$

Now consider the following combination

$$\sum_{i=1}^{6}\omega^i M_i=\omega M_1+\omega^2 M_2+\omega^3 M_3+\omega^4 M_4+\omega^5 M_5+\omega^6 M_6$$

$$=\omega\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}+\omega^2\begin{bmatrix}0&0&0&1&0&0\\1&0&0&0&0&0\\0&0&0&0&1&0\\0&1&0&0&0&0\\0&0&0&0&1&0\\0&0&1&0&0&0\end{bmatrix}$$

$$+\omega^3\begin{bmatrix}0&0&0&0&1&0\\0&0&1&0&0&0\\1&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&1&0&0\\0&1&0&0&0&0\end{bmatrix}+\omega^4\begin{bmatrix}0&1&0&0&0&0\\0&0&0&1&0&0\\0&0&0&0&0&1\\1&0&0&0&0&0\\0&0&1&0&0&0\\0&0&0&0&1&0\end{bmatrix}$$

$$+\omega^5\begin{bmatrix}0&0&1&0&0&0\\0&0&0&0&0&1\\0&1&0&0&0&0\\0&0&0&0&1&0\\1&0&0&0&0&0\\0&0&0&1&0&0\end{bmatrix}+\omega^6\begin{bmatrix}0&0&0&0&0&1\\0&0&0&0&1&0\\0&0&0&1&0&0\\0&0&1&0&0&0\\0&1&0&0&0&0\\1&0&0&0&0&0\end{bmatrix}$$

$$=\begin{bmatrix}\omega&\omega^4&\omega^5&\omega^2&\omega^3&\omega^6\\\omega^2&\omega&\omega^3&\omega^4&\omega^6&\omega^5\\\omega^3&\omega^5&\omega&\omega^6&\omega^2&\omega^4\\\omega^4&\omega^2&\omega^6&\omega&\omega^5&\omega^3\\\omega^5&\omega^6&\omega^4&\omega^3&\omega&\omega^2\\\omega^6&\omega^3&\omega^2&\omega^5&\omega^4&\omega\end{bmatrix}=Q'(Say)$$

This matrix Q' is equivalent to matrix Q of Fourier matrix F_7 as by the following elementary operations we get Q of F_7

$$C_2\leftrightarrow C_4 \text{ and } C_3\leftrightarrow C_5$$

III. CONCLUSION

From the above results we conclude that Fourier matrices of order 3, 5 and 7 can be constructed with the help of suitable permutation groups. Also these results give some insight about the type of permutation groups that are needed for the construction of Fourier matrices of prime order.

IV. FUTURE WORK

The above constructions can be generalized of all prime order p.

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