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A Study On Conjugate Normal Bimatrices

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Abstract: The characterization of conjugate normal bimatrices are given as a generalization of the results for conjugate normal matrices.

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I. Introduction

Let $C_{n\times n}$ be the space of $n\times n$ complex matrices of order n. Let C_n be the space of all complex n-tuples. A matrix $A_B = A_1 \cup A_2$ is called a bimatrix if A_1 and A_2 are matrices of same or different orders [3]. We consider here only matrices of same order. A bimatrix A_B is called normal if $A_B A_B^* = A_B^* A_B$ [4,5]. The class of normal matrices important throughout matrix analysis it is especially important in matters related to similarity transformations and even more specifically to unitary similarity transformations such significant matrix classes as Hermitian, Skew hermitian and Unitary Matrices are subclasses of Normal matrices. Conjugate normal matrices play the same important role in the theory of unitary congruence as the conventional normal matrices do with respect to unitary similarities[8]. However, unlike the latter, the properties of conjugate normal matrices are not widely known. Motivated by this fact ,we give a survey of the properties of these matrices.

II .Conjugate Normal Bimatrix

In this section some important results of conjugate normal matrix is found in [1,2,6,7] generalized to conjugate normal bimatrix.

Definition 2.1

A bimatrix $A_B \in C_{n\times n}$ is said to be conjugate normal if $A_B A_B^* = \overline{A_B^* A_B}$.

That is , $A_1 A_1^* \cup A_2 A_2^* = \overline{A_1^* A_1} \cup \overline{A_2^* A_2}$.

That is, $A_B A_B^* = A_B^T \overline{A}_B \text{(or)} A_B^* A_B = \overline{A}_B A_B^T$.

Example 2.2

Let

$$A_B = A_1 \cup A_2 = \begin{bmatrix} 1 & 1+i \\ 1+i & 0 \end{bmatrix} \cup \begin{bmatrix} 1 & 3+5i \\ 3+5i & 0 \end{bmatrix}$$

$$A_B A_B^* = \begin{bmatrix} 3 & 1-i \\ 1+i & 2 \end{bmatrix} \cup \begin{bmatrix} 35 & 3-5i \\ 3+5i & 34 \end{bmatrix} \quad (1)$$

$$\overline{A_B A_B^*} = \begin{bmatrix} 3 & 1-i \\ 1+i & 2 \end{bmatrix} \cup \begin{bmatrix} 35 & 3-5i \\ 3+5i & 34 \end{bmatrix} \quad (2)$$

From (1) and (2), we get $A_B A_B^* = \overline{A_B^* A_B}$.

Hence, A_B is a conjugate normal bimatrix.

For any bimatrix $A_B \in C_{n\times n}$, we can write $A_B = S_B + K_B$,such that S_B is symmetric and K_B is skew-symmetric. This decomposition for bimatrix is A_B uniquely determined by $S_B = \frac{1}{2}\left(A_B + A_B^*\right)$ and $K_B = \frac{1}{2}\left(A_B - A_B^*\right)$.

We introduce the bimatrices $A_{BL} = \overline{A}_B A_B$ and $A_{BR} = A_B \overline{A}_B = \overline{A}_{BL}$.

Theorem 2.3

If A_B is a conjugate normal bimatrix then A_{BL} and A_{BR} are normal bimatrix.

Proof

Given that A_B is a conjugate normal bimatrix.

That is $A_B A_B^* = \overline{A_B^* A_B}$.

That is , $A_B A_B^* = A_B^T \overline{A}_B \text{(or)} A_B^* A_B = \overline{A}_B A_B^T$

We have $A_{BR} A_{BR}^* = \left(A_B \overline{A}_B\right)\left(A_B \overline{A}_B\right)^*$

$$= \left[\left(A_1 \cup A_2\right)\left(\overline{A}_1 \cup \overline{A}_2\right)\right]\left[\left(A_1 \cup A_2\right)\left(\overline{A}_1 \cup \overline{A}_2\right)\right]^*$$

$$= \left(A_1\overline{A}_1 \cup A_2\overline{A}_2\right)\left(A_1\overline{A}_1 \cup A_2\overline{A}_2\right)^*$$

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$$=\left(A_1\bar{A}_1\cup A_2\bar{A}_2\right)\left(\left(A_1\bar{A}_1\right)^*\cup\left(A_2\bar{A}_2\right)^*\right)$$

$$=\left(A_1\bar{A}_1\cup A_2\bar{A}_2\right)\left[\left(\bar{A}_1\right)^* A_1^*\cup\left(\bar{A}_2\right)^* A_2^*\right]$$

$$=\left(A_1\bar{A}_1\left(\bar{A}_1\right)^* A_1^*\right)\cup\left(A_2\bar{A}_2\left(\bar{A}_2\right)^* A_2^*\right)$$

$$=\left(A_1\left(\bar{A}_1A_1^T\right)A_1^*\right)\cup\left(A_2\left(\bar{A}_2A_2^T\right)A_2^*\right)$$

$$=\left(A_1A_1^*\right)\left(A_1A_1^*\right)\cup\left(A_2A_2^*\right)\left(A_2A_2^*\right)$$

$$=\left(A_1A_1^*\right)^2\cup\left(A_2A_2^*\right)^2=\left(A_1A_1^*\cup A_2A_2^*\right)^2$$

$$=\left[\left(A_1\cup A_2\right)\left(A_1^*\cup A_2^*\right)\right]^2 \quad A_{BR}A_{BR}^*=\left(A_BA_B^*\right)^2$$

(3)

$$A_{BR}^*A_{BR}=\left(A_B\bar{A}_B\right)^*\left(A_B\bar{A}_B\right)$$

$$=\left[\left(A_1\cup A_2\right)\left(\bar{A}_1\cup\bar{A}_2\right)\right]^*\left[\left(A_1\cup A_2\right)\left(\bar{A}_1\cup\bar{A}_2\right)\right]$$

$$=\left(A_1\bar{A}_1\cup A_2\bar{A}_2\right)^*\left(A_1\bar{A}_1\cup A_2\bar{A}_2\right)$$

$$=\left[\left(A_1\bar{A}_1\right)^*\cup\left(A_2\bar{A}_2\right)^*\right]\left(A_1\bar{A}_1\cup A_2\bar{A}_2\right)$$

$$=\left[\left(\left(\bar{A}_1\right)^* A_1^*\right)\cup\left(\left(\bar{A}_2\right)^* A_2^*\right)\right]\left(A_1\bar{A}_1\cup A_2\bar{A}_2\right)$$

$$=\left(A_1^TA_1^*\cup A_2^TA_2^*\right)\left(A_1\bar{A}_1\cup A_2\bar{A}_2\right)$$

$$=\left(A_1^T\left(A_1^*A_1\right)\bar{A}_1\right)\cup\left(A_2^T\left(A_2^*A_2\right)\bar{A}_2\right)$$

$$=\left(A_1^T\bar{A}_1\right)\left(A_1^T\bar{A}_1\right)\cup\left(A_2^T\bar{A}_2\right)\left(A_2^T\bar{A}_2\right)$$

$$=\left(A_1A_1^*\right)\left(A_1A_1^*\right)\cup\left(A_2A_2^*\right)\left(A_2A_2^*\right)$$

$$=\left(A_1A_1^*\right)^2\cup\left(A_2A_2^*\right)^2$$

$$=\left(A_1A_1^*\cup A_2A_2^*\right)^2$$

$$=\left[\left(A_1\cup A_2\right)\left(A_1^*\cup A_2^*\right)\right]^2$$

$$A_{BR}^*A_{BR}=\left(A_BA_B^*\right)^2 \qquad (4)$$

From (3) and (4), we get

$$A_{BR}A_{BR}^*=A_{BR}^*A_{BR}.$$

Thus, A_{BR} is normal.

Hence, $A_{BL}=\bar{A}_{BR}$ is normal as well.

Remark 2.4

The reverse implication of the above theorem is false. To state the next proposition, we associate with each bimatrix $A_B\in C_{n\times n}$ the bimatrix $\hat{A}_B=\begin{bmatrix}O_B & A_B\\ \bar{A}_B & O_B\end{bmatrix}$.

Theorem 2.5

A bimatrix $A_B\in C_{n\times n}$ is conjugate normal if and only if $\hat{A}_B$ is normal.

Proof

Let $\hat{A}_B=\begin{bmatrix}O_B & A_B\\ \bar{A}_B & O_B\end{bmatrix}=\begin{bmatrix}O_1\cup O_2 & A_1\cup A_2\\ \bar{A}_1\cup\bar{A}_2 & O_1\cup O_2\end{bmatrix}$

$$\left(\hat{A}_B\right)^*=\begin{bmatrix}O_1\cup O_2 & A_1\cup A_2\\ \bar{A}_1\cup\bar{A}_2 & O_1\cup O_2\end{bmatrix}^*$$

$$=\begin{bmatrix}O_1\cup O_2 & \bar{A}_1\cup\bar{A}_2\\ \left(A_1\right)^T\cup\left(A_2\right)^T & O_1\cup O_2\end{bmatrix}$$

$$\left(\hat{A}_B\right)^*=\begin{bmatrix}O_1\cup O_2 & A_1^T\cup A_2^T\\ A_1^*\cup A_2^* & O_1\cup O_2\end{bmatrix}$$

$$\left(\hat{A}_B\right)\left(\hat{A}_B\right)^*=\begin{bmatrix}O_1\cup O_2 & A_1\cup A_2\\ \bar{A}_1\cup\bar{A}_2 & O_1\cup O_2\end{bmatrix}\begin{bmatrix}O_1\cup O_2 & A_1^T\cup A_2^T\\ A_1^*\cup A_2^* & O_1\cup O_2\end{bmatrix}$$

$$=\begin{bmatrix}A_1A_1^*\cup A_2A_2^* & O_1\cup O_2\\ O_1\cup O_2 & \bar{A}_1A_1^T\cup\bar{A}_2A_2^T\end{bmatrix}$$

$$=\begin{bmatrix}\left(A_1\cup A_2\right)\left(A_1^*\cup A_2^*\right) & O_1\cup O_2\\ O_1\cup O_2 & \left(\bar{A}_1\cup\bar{A}_2\right)\left(A_1^T\cup A_2^T\right)\end{bmatrix}$$

$$=\begin{bmatrix}A_BA_B^* & O_B\\ O_B & \bar{A}_BA_B^T\end{bmatrix}$$

$$\left(\hat{A}_B\right)\left(\hat{A}_B\right)^*=A_BA_B^*\oplus\bar{A}_BA_B^T \qquad (5)$$

$$\left(\hat{A}_B\right)^*\left(\hat{A}_B\right)=\begin{bmatrix}O_1\cup O_2 & A_1^T\cup A_2^T\\ A_1^*\cup A_2^* & O_1\cup O_2\end{bmatrix}\begin{bmatrix}O_1\cup O_2 & A_1\cup A_2\\ \bar{A}_1\cup\bar{A}_2 & O_1\cup O_2\end{bmatrix}$$

$$=\begin{bmatrix}A_1^T\bar{A}_1\cup A_2^T\bar{A}_2 & O_1\cup O_2\\ O_1\cup O_2 & A_1^*A_1\cup A_2^*A_2\end{bmatrix}$$

$$=\begin{bmatrix}\left(A_1^T\cup A_2^T\right)\left(\bar{A}_1\cup\bar{A}_2\right) & O_1\cup O_2\\ O_1\cup O_2 & \left(A_1^*\cup A_2^*\right)\left(A_1\cup A_2\right)\end{bmatrix}$$

$$=\begin{bmatrix}A_B^T\bar{A}_B & O_B\\ O_B & A_B^*A_B\end{bmatrix}$$

$$\left(\hat{A}_B\right)^*\left(\hat{A}_B\right)=A_B^T\bar{A}_B\oplus A_B^*A_B. \qquad (6)$$

From (5) and (6), we get

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$$\left(\hat{A}_B\right)\left(\hat{A}_B\right)^* = \left(\hat{A}_B\right)^*\left(\hat{A}_B\right)$$

Hence, $\hat{A}_B$ is normal.

In the following theorem im(.) and ker(.) denote the range and the null space of the corresponding bimatrix.

Theorem 2.6

If $A_B \in C_{n\times n}$ is conjugate normal, then $im\left(A_B\right) = im\left(A_B^T\right)$ and $\ker\left(A_B\right) = \ker\left(A_B^T\right)$.

Proof

For any bimatrix A_B, we have

$im\left(A_B A_B^*\right) = im\left(A_B\right)$ and

$\ker\left(A_B A_B^*\right) = \ker\left(A_B^*\right)$.

So $im\left(A_B\right) = im\left(A_B A_B^*\right)$

$= im\left(\left(A_1 \cup A_2\right)\left(A_1^* \cup A_2^*\right)\right)$

$= im\left(A_1 A_1^* \cup A_2 A_2^*\right)$

$= im\left(A_1^T \bar{A}_1 \cup A_2^T \bar{A}_2\right)$

$= im\left(A_1^T\left(A_1^T\right)^* \cup A_2^T\left(A_2^T\right)^*\right)$

$= im\left(A_1^T \cup A_2^T\right) = im\left(\left(A_1 \cup A_2\right)^T\right)$

$= imt\left(A_B^T\right)$

Hence, $im\left(A_B\right) = im\left(A_B^T\right)$.

$\ker\left(A_B^*\right) = \ker\left(A_B A_B^*\right)$

$= \ker\left(\left(A_1 \cup A_2\right)\left(A_1^* \cup A_2^*\right)\right)$

$= \ker\left(A_1 A_1^* \cup A_2 A_2^*\right)$

$= \ker\left(A_1^T \bar{A}_1 \cup A_2^T \bar{A}_2\right)$

$\ker\left(A_B^*\right) = \ker\left(\bar{A}_1 \cup \bar{A}_2\right)$

$\ker\left(\left(A_B^*\right)^*\right) = \ker\left(\left(\bar{A}_1 \cup \bar{A}_2\right)^*\right)$

$\ker A_B = \ker\left(\left(\overline{A_1^*}\right) \cup \left(\overline{A_2^*}\right)\right) = \ker\left(A_1^T \cup A_2^T\right)$

Hence, $\ker\left(A_B\right) = \ker\left(A_B^T\right)$.

Theorem 2.7

Let $A_B \in C_{n\times n}$ be a bimatrix. The following statements are equivalent.

(i) A_B is a conjugate normla bimatrix.

(ii) $\bar{A}_B$ is a conjugate normal bimatrix.

(iii) A_B^T is a conjugate normal bimatrix.

(iv) A_B^* is a conjugate normal bimatrix.

(v) A_B^{-1} is a conjugate normal bimatrix, if A_B^{-1} exists.

(vi) A^+ is a conjugate normal bimatrix.

(vii) λA_B is a conjugate normal Bimatrix, Where λ is a real numbers.

Proof of $(i) \Leftrightarrow (ii)$

A_B is a cojugate normal bimatrix

$\Leftrightarrow A_B A_B^* = \overline{A_B^* A_B}$

$\Leftrightarrow \left(A_1 \cup A_2\right)\left(A_1^* \cup A_2^*\right) = \overline{\left(A_1^* \cup A_2^*\right)\left(A_1 \cup A_2\right)} \Leftrightarrow A_1 A_1^* \cup A_2 A_2^* = \overline{\left(A_1^* A_1 \cup A_2^* A_2\right)}$

$\Leftrightarrow A_1 A_1^* \cup A_2 A_2^* = \overline{A_1^* A_1} \cup \overline{A_2^* A_2}$

$\Leftrightarrow A_1 A_1^* \cup A_2 A_2^* = \overline{A_1^*}\bar{A}_1 \cup \overline{A_2^*}\bar{A}_2$

$\Leftrightarrow A_1 A_1^* \cup A_2 A_2^* = A_1^T \bar{A}_1 \cup A_2^T \bar{A}_2$

$\Leftrightarrow \left(\overline{A_1 A_1^* \cup A_2 A_2^*}\right) = \left(\overline{A_1^T \bar{A}_1 \cup A_2^T \bar{A}_2}\right)$

$\Leftrightarrow \overline{A_1 A_1^*} \cup \overline{A_2 A_2^*} = \overline{A_1^T \bar{A}_1} \cup \overline{A_2^T \bar{A}_2}$

$\Leftrightarrow \bar{A}_1 \overline{A_1^*} \cup \bar{A}_2 \overline{A_2^*} = \overline{A_1^T} A_1 \cup \overline{A_2^T} A_2$

$\Leftrightarrow \bar{A}_1 A_1^T \cup \bar{A}_2 A_2^T = A_1^* A_1 \cup A_2^* A_2$

$\Leftrightarrow \left(\bar{A}_1 \cup \bar{A}_2\right)\left(A_1^T \cup A_2^T\right)$

$= \left(A_1^* \cup A_2^*\right)\left(A_1 \cup A_2\right)$

$\Leftrightarrow \bar{A}_B A_B^T = A_B^* A_B$

$\Leftrightarrow \bar{A}_B$ is a conjugate normal bimatrix.

Proof of $(i) \Leftrightarrow (iii)$

A_B is a conjugate normal bimatrix.

$\Leftrightarrow A_B A_B^* = \overline{A_B^* A_B}$

$\Leftrightarrow \left(A_1 \cup A_2\right)\left(A_1^* \cup A_2^*\right)$

$= \overline{\left(A_1^* \cup A_2^*\right)\left(A_1 \cup A_2\right)}$

$\Leftrightarrow A_1 A_1^* \cup A_2 A_2^* = \overline{\left(A_1^* A_1 \cup A_2^* A_2\right)}$

$\Leftrightarrow A_1 A_1^* \cup A_2 A_2^* = \overline{A_1^* A_1} \cup \overline{A_2^* A_2}$

$\Leftrightarrow A_1 A_1^* \cup A_2 A_2^* = \left(\overline{A_1^*}\right)\bar{A}_1 \cup \left(\overline{A_2^*}\right)\bar{A}_2$

$\Leftrightarrow A_1 A_1^* \cup A_2 A_2^* = A_1^T \bar{A}_1 \cup A_2^T \bar{A}_2$

$\Leftrightarrow \left(A_1 A_1^* \cup A_2 A_2^*\right)^T = \left(A_1^T \bar{A}_1 \cup A_2^T \bar{A}_2\right)^T$

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$$\Leftrightarrow \left(A_1A_1^*\right)^T \cup \left(A_2A_2^*\right)^T$$

$$= \left(A_1^T\bar{A}_1\right)^T \cup \left(A_2^T\bar{A}_2\right)^T$$

$$\Leftrightarrow \left(A_1^*\right)^T A_1^T \cup \left(A_2^*\right)^T A_2^T$$

$$= \left(\bar{A}_1\right)^T A_1 \cup \left(\bar{A}_2\right)^T A_2$$

$$\Leftrightarrow \bar{A}_1A_1^T \cup \bar{A}_2A_2^T = A_1^*A_1 \cup A_2^*A_2$$

$$\Leftrightarrow \left(\bar{A}_1 \cup \bar{A}_2\right)\left(A_1^T \cup A_2^T\right)$$

$$= \left(A_1^* \cup A_2^*\right)\left(A_1 \cup A_2\right)$$

$$\Leftrightarrow \bar{A}_BA_B^T = A_B^*A_B$$

$\Leftrightarrow A_B^T$ is a conjugate normal bimatrix.

Proof of $(i) \Leftrightarrow (iv)$

A_B is a conjugate normal bimatrix.

$$\Leftrightarrow A_BA_B^* = \overline{A_B^*A_B}$$

$$\Leftrightarrow \left(A_1 \cup A_2\right)\left(A_1^* \cup A_2^*\right)$$

$$= \overline{\left(A_1^* \cup A_2^*\right)\left(A_1 \cup A_2\right)}$$

$$\Leftrightarrow A_1A_1^* \cup A_2A_2^* = \overline{\left(A_1^*A_1 \cup A_2^*A_2\right)}$$

$$\Leftrightarrow A_1A_1^* \cup A_2A_2^* = \overline{A_1^*A_1} \cup \overline{A_2^*A_2}$$

$$\Leftrightarrow A_1A_1^* \cup A_2A_2^* = \overline{A_1^*}\bar{A}_1 \cup \overline{A_2^*}\bar{A}_2$$

$$\Leftrightarrow A_1A_1^* \cup A_2A_2^* = A_1^T\bar{A}_1 \cup A_2^T\bar{A}_2$$

$$\Leftrightarrow \left(A_1A_1^* \cup A_2A_2^*\right)^* = \left(A_1^T\bar{A}_1 \cup A_2^T\bar{A}_2\right)^*$$

$$\Leftrightarrow \left(A_1A_1^*\right)^* \cup \left(A_2A_2^*\right)^*$$

$$= \left(A_1^T\bar{A}_1\right)^* \cup \left(A_2^T\bar{A}_2\right)^*$$

$$\Leftrightarrow A_1A_1^* \cup A_2A_2^*$$

$$= \left(\bar{A}_1\right)^*\left(A_1^T\right)^* \cup \left(\bar{A}_1\right)^*\left(A_2^T\right)^*$$

$$\Leftrightarrow A_1A_1^* \cup A_2A_2^* = \left(\overline{A_1^*}\right)\bar{A}_1 \cup \left(\overline{A_2^*}\right)\bar{A}_2$$

$$\Leftrightarrow A_1A_1^* \cup A_2A_2^* = A_1^T\bar{A}_1 \cup A_2^T\bar{A}_2$$

$$\Leftrightarrow \left(A_1A_1^* \cup A_2A_2^*\right)^+ = \left(A_1^T\bar{A}_1 \cup A_2^T\bar{A}_2\right)^+$$

$$\Leftrightarrow \left(A_1A_1^*\right)^+ \cup \left(A_2A_2^*\right)^+$$

$$= \left(A_1^T\bar{A}_1\right)^+ \cup \left(A_2^T\bar{A}_2\right)^+$$

$$\Leftrightarrow \left(A_1^*\right)^+ A_1^+ \cup \left(A_2^*\right)^+ A_2^+$$

$$= \left(\bar{A}_1\right)^+\left(A_1^T\right)^+ \cup \left(\bar{A}_2\right)^+\left(A_2^T\right)^+$$

$$\Leftrightarrow \left(A_1^+\right)^* A_1^+ \cup \left(A_2^+\right)^* A_2^+$$

$$= \left(\overline{A_1^+}\right)\left(A_1^+\right)^T \cup \left(\overline{A_2^+}\right)\left(A_2^+\right)^T$$

$$\Leftrightarrow \left(\left(A_1^+\right)^* \cup \left(A_2^+\right)^*\right)\left(A_1^+ \cup A_2^+\right)$$

$$= \left(\left(\overline{A_1^+}\right) \cup \left(A_2^+\right)^T\right)\left(\left(A_1^+\right)^T \cup \left(A_2^+\right)^T\right)$$

$$\Leftrightarrow \left(A_B^+\right)^* A_B^+ = \left(\overline{A_B^+}\right)\left(A_B^+\right)^T$$

$\Leftrightarrow A_B^+$ is a conjugate normal bimatrix.

Proof of $(i) \Leftrightarrow (vii)$

A_B is a conjugate normal bimatrix.

$$\Leftrightarrow A_BA_B^* = \overline{A_B^*A_B}$$

$$\Leftrightarrow \left(A_1 \cup A_2\right)\left(A_1^* \cup A_2^*\right)$$

$$= \overline{\left(A_1^* \cup A_2^*\right)\left(A_1 \cup A_2\right)}$$

$$\Leftrightarrow A_1A_1^* \cup A_2A_2^* = \overline{\left(A_1^*A_1 \cup A_2^*A_2\right)}$$

$$\Leftrightarrow A_1A_1^* \cup A_2A_2^* = \overline{A_1^*A_1} \cup \overline{A_2^*A_2}$$

$$\Leftrightarrow A_1A_1^* \cup A_2A_2^* = \left(\overline{A_1^*}\right)\bar{A}_1 \cup \left(\overline{A_2^*}\right)\bar{A}_2$$

$$\Leftrightarrow A_1A_1^* \cup A_2A_2^* = A_1^T\bar{A}_1 \cup A_2^T\bar{A}_2$$

$$\Leftrightarrow \lambda^2\left(A_1A_1^* \cup A_2A_2^*\right) = \lambda^2\left(A_1^T\bar{A}_1 \cup A_2^T\bar{A}_2\right)$$

$$\Leftrightarrow \lambda\left(A_1 \cup A_2\right)\lambda\left(A_1^* \cup A_2^*\right)$$

$$= \lambda\left(A_1^T \cup A_2^T\right)\lambda\left(\bar{A}_1 \cup \bar{A}_2\right)$$

$$\Leftrightarrow \lambda\left(A_B\right)\lambda\left(A_B^*\right) = \lambda\left(A_B^T\right)\lambda\left(\bar{A}_B\right)$$

$$\Leftrightarrow \left(\lambda A_B\right)\left(\lambda A_B\right)^* = \left(\lambda A_B\right)^T\left(\overline{\lambda A_B}\right)$$

$\Leftrightarrow \lambda A_B$ is a conjugate normal bimatrix.

$$\Leftrightarrow A_1A_1^* \cup A_2A_2^* = A_1^T\bar{A}_1 \cup A_2^T\bar{A}_2$$

$$\Leftrightarrow \left(A_1 \cup A_2\right)\left(A_1^* \cup A_2^*\right)$$

$$= \left(A_1^T \cup A_2^T\right)\left(\bar{A}_1 \cup \bar{A}_2\right)$$

$$\Leftrightarrow A_BA_B^* = A_B^T\bar{A}_B$$

$\Leftrightarrow A_B^*$ is a conjugate normal bimatrix.

Proof of $(i) \Leftrightarrow (v)$

A_B is a conjugate normal bimatrix.

$$\Leftrightarrow A_BA_B^* = \overline{A_B^*A_B}$$

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$$\Leftrightarrow (A_1 \cup A_2)(A_1^* \cup A_2^*) = \overline{(A_1^* \cup A_2^*)(A_1 \cup A_2)}$$

$$\Leftrightarrow A_1A_1^* \cup A_2A_2^* = \overline{(A_1^*A_1 \cup A_2^*A_2)}$$

$$\Leftrightarrow A_1A_1^* \cup A_2A_2^* = \overline{A_1^*A_1} \cup \overline{A_2^*A_2}$$

$$\Leftrightarrow A_1A_1^* \cup A_2A_2^* = \left(\overline{A_1^*}\right)\bar{A}_1 \cup \left(\overline{A_2^*}\right)\bar{A}_2$$

$$\Leftrightarrow A_1A_1^* \cup A_2A_2^* = A_1^T\bar{A}_1 \cup A_2^T\bar{A}_2$$

$$\Leftrightarrow (A_1A_1^* \cup A_2A_2^*)^{-1} = (A_1^T\bar{A}_1 \cup A_2^T\bar{A}_2)^{-1}$$

$$\Leftrightarrow (A_1A_1^*)^{-1} \cup (A_2A_2^*)^{-1} = (A_1^T\bar{A}_1)^{-1} \cup (A_2^T\bar{A}_2)^{-1}$$

$$\Leftrightarrow (A_1^*)^{-1}A_1^{-1} \cup (A_2^*)^{-1}A_2^{-1} = (\bar{A}_1)^{-1}(A_1^T)^{-1} \cup (\bar{A}_2)^{-1}(A_2^T)^{-1}$$

$$\Leftrightarrow (A_1^{-1})^*A_1^{-1} \cup (A_2^{-1})^*A_2^{-1} = \left(\overline{A_1^{-1}}\right)(A_1^{-1})^T \cup \left(\overline{A_2^{-1}}\right)(A_2^{-1})^T$$

$$\Leftrightarrow \left((A_1^{-1})^* \cup (A_2^{-1})^*\right)(A_1^{-1} \cup A_2^{-1}) = \left(\left(\overline{A_1^{-1}}\right) \cup \left(\overline{A_2^{-1}}\right)\right)\left((A_1^{-1})^T \cup (A_2^{-1})^T\right) \Leftrightarrow$$

$$(A_B^{-1})^*A_B^{-1} = \left(\overline{A_B^{-1}}\right)(A_B^{-1})^T$$

$\Leftrightarrow A_B^{-1}$ is a conjugate normal bimatrix if A_B^{-1} exists.

Proof of $(i) \Leftrightarrow (vi)$

A_B is a conjugate normal bimatrix.

$$\Leftrightarrow A_BA_B^* = \overline{A_B^*A_B}$$

$$\Leftrightarrow (A_1 \cup A_2)(A_1^* \cup A_2^*) = \overline{(A_1^* \cup A_2^*)(A_1 \cup A_2)}$$

$$\Leftrightarrow A_1A_1^* \cup A_2A_2^* = \overline{(A_1^*A_1 \cup A_2^*A_2)}$$

$$\Leftrightarrow A_1A_1^* \cup A_2A_2^* = \overline{A_1^*A_1} \cup \overline{A_2^*A_2}$$

Theorem 2.8

Let $A_B \in C_{n\times n}$ be a bimatrix.

(i) If A_B is a conjugate normal bimatrix, then iA_B is a conjugate normal bimatrix.

(ii) If A_B is a conjugate normal bimatrix, then $-iA_B$ is a conjugate normal bimatrix.

Proof of (i)

Given that A_B is a conjugate normal bimatrix. That is , $A_BA_B^* = \overline{A_B^*A_B}$

$$(A_1 \cup A_2)(A_1^* \cup A_2^*) = \overline{(A_1^* \cup A_2^*)(A_1 \cup A_2)}$$

$$A_1A_1^* \cup A_2A_2^* = \overline{(A_1^*A_1 \cup A_2^*A_2)}$$

$$A_1A_1^* \cup A_2A_2^* = \overline{A_1^*A_1} \cup \overline{A_2^*A_2}$$

$$A_1A_1^* \cup A_2A_2^* = \left(\overline{A_1^*}\right)\bar{A}_1 \cup \left(\overline{A_2^*}\right)\bar{A}_2$$

$$A_1A_1^* \cup A_2A_2^* = A_1^T\bar{A}_1 \cup A_2^T\bar{A}_2$$

$$i^2(A_1A_1^* \cup A_2A_2^*) = i^2(A_1^T\bar{A}_1 \cup A_2^T\bar{A}_2)$$

$$i(A_1 \cup A_2)\left((i)(A_1 \cup A_2)\right)^* = i(A_1^T \cup A_2^T)\left(i\left(\overline{A_1 \cup A_2}\right)\right)$$

$$(iA_B)\left(-(iA_B)^*\right) = (iA_B)^T\left(-\left(i\overline{A_B}\right)\right)$$

$$(iA_B)(iA_B)^* = (iA_B)^T\left(\overline{iA_B}\right)$$

Hence, iA_B is a conjugate normal bimatrix.

Similarly, we can prove that $-iA_B$ is a conjugate normal bimatrix.

Theorem 2.9

If $A_B \in C_{n\times n}$ be a conjugate normal bimatrix, then $A_B\bar{A}_B$ and $\bar{A}_BA_B$ are normal bimatrices.

Proof

Given that A_B is a conjugate normal bimatrix. That is,

$$A_BA_B^* = \overline{A_B^*A_B}$$

That is , $A_BA_B^* = A_B^T\overline{A_B}$.

Now

$$(A_B\bar{A}_B)(A_B\bar{A}_B)^* = (A_1 \cup A_2)\left(\overline{A_1 \cup A_2}\right)\left[(A_1 \cup A_2)\left(\overline{A_1 \cup A_2}\right)\right]^*$$

$$= (A_1 \cup A_2)(\bar{A}_1 \cup \bar{A}_2)\left[(A_1 \cup A_2)(\bar{A}_1 \cup \bar{A}_2)\right]^*$$

$$= (A_1\bar{A}_1 \cup A_2\bar{A}_2)(A_1\bar{A}_1 \cup A_2\bar{A}_2)^*$$

$$= (A_1\bar{A}_1 \cup A_2\bar{A}_2)\left[(A_1\bar{A}_1)^* \cup (A_2\bar{A}_2)^*\right]$$

$$= (A_1\bar{A}_1 \cup A_2\bar{A}_2)\left[(\bar{A}_1)^*A_1^* \cup (\bar{A}_2)^*A_2^*\right]$$

$$= (A_1\bar{A}_1 \cup A_2\bar{A}_2)\left[\left(\overline{A_1^*}\right)A_1^* \cup \left(\overline{A_2^*}\right)(A_2^*)\right]$$

$$= (A_1\bar{A}_1 \cup A_2\bar{A}_2)(A_1^TA_1^* \cup A_1^TA_2^*)$$

$$= (A_1\bar{A}_1A_1^TA_1^*) \cup (A_2\bar{A}_2A_2^TA_2^*)$$

$$= \left[A_1(\bar{A}_1A_1^T)A_1^*\right] \cup \left[A_2(\bar{A}_2A_2^T)A_2^*\right]$$

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$$=\left(A_1\cup A_2\right)\left(\bar{A}_1A_1^T\cup\bar{A}_2A_2^T\right)\left(A_1^*\cup A_2^*\right)$$
$$=\left(A_1\cup A_2\right)\left[\left(\bar{A}_1\cup\bar{A}_2\right)\left(A_1^T\cup A_2^T\right)\right]\left(A_1\cup A_2\right)^*$$
$$=A_B\left(\bar{A}_BA_B^T\right)A_B^*$$
$$=A_B\left(A_B^*A_B\right)A_B^*$$
$$=\left(A_BA_B^*\right)\left(A_BA_B^*\right)$$
$$\left(A_B\bar{A}_B\right)\left(A_B\bar{A}_B\right)^*=\left(A_BA_B^*\right)^2 \qquad (7)$$

Now

$$\left(A_B\bar{A}_B\right)^*\left(A_B\bar{A}_B\right)=\left[\left(A_1\cup A_2\right)\left(\overline{A_1\cup A_2}\right)\right]^*\left(A_1\cup A_2\right)\left(\overline{A_1\cup A_2}\right)$$
$$=\left[\left(A_1\cup A_2\right)\left(\bar{A}_1\cup\bar{A}_2\right)\right]^*\left(A_1\cup A_2\right)\left(\bar{A}_1\cup\bar{A}_2\right)$$
$$=\left(A_1\bar{A}_1\cup A_2\bar{A}_2\right)^*\left(A_1\bar{A}_1\cup A_2\bar{A}_2\right)$$
$$=\left[\left(A_1\bar{A}_1\right)^*\cup\left(A_2\bar{A}_2\right)^*\right]\left(A_1\bar{A}_1\cup A_2\bar{A}_2\right)$$
$$=\left(\left(\bar{A}_1\right)^*A_1^*\cup\left(\bar{A}_2\right)^*A_2^*\right)\left(A_1\bar{A}_1\cup A_2\bar{A}_2\right)$$
$$=\left[\left(\overline{A_1^*}\right)A_1^*\cup\left(\overline{A_2^*}\right)A_2^*\right]\left(A_1\bar{A}_1\cup A_2\bar{A}_2\right)$$
$$=\left(A_1^TA_1^*\cup A_2^TA_2^*\right)\cup\left(A_1\bar{A}_1\cup A_2\bar{A}_2\right)$$
$$=\left(A_1^TA_1^*A_1\bar{A}_1\right)\cup\left(A_2^TA_2^*A_2\bar{A}_2\right)$$
$$=\left(A_1^TA_1^*\bar{A}_1A_1\right)\cup\left(A_2^TA_2^*\bar{A}_2A_2\right)$$
$$=\left(\left(A_1^T\bar{A}_1\right)A_1^*A_1\right)\cup\left(\left(A_2^T\bar{A}_2\right)A_2^*A_2\right)$$
$$=\left(A_1^T\bar{A}_1\cup A_2^T\bar{A}_2\right)\left(A_1^*\cup A_2^*\right)\left(A_1\cup A_2\right)$$
$$=\left[\left(A_1^T\cup A_2^T\right)\left(\bar{A}_1\cup\bar{A}_2\right)\right]\left(A_1^*\cup A_2^*\right)\left(A_1\cup A_2\right)$$
$$=\left(A_B^T\bar{A}_B\right)A_B^*A_B$$
$$=\left(A_BA_B^*\right)A_BA_B^*$$
$$\left(A_B\bar{A}_B\right)^*\left(A_B\bar{A}_B\right)=\left(A_BA_B^*\right)^2 \qquad (8)$$

From (7) and (8), we get

$$\left(A_B\bar{A}_B\right)\left(A_B\bar{A}_B\right)^*=\left(A_B\bar{A}_B\right)^*\left(A_B\bar{A}_B\right).$$

Hence, $A_B\bar{A}_B$ is a normal bimatrix.

Similarly, we can prove that $\bar{A}_BA_B$ is a normal bimatrix.

III. CONCLUSION

The concept of conjugate normal bimatrices are studied as a generalization of conjugate normal matrices. Some of the properties of conjugate normal matrices are extended to conjugate normal bimatrices. Other properties of conjugate normal matrices can be extended to conjugate normal bimatrices as a future study.

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