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Analysis of Impulsive Prey-Predator Fish Harvesting Mathematical Model

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Abstract: Bio-economic modeling of biological resources such as fisheries and stock breedings have attained a great attention in the past few years. Unregulated and extensive harvesting of fishes may result in the extinction of several fish species. So the optimal harvesting management of fishery resources is a need of today. We propose a prey-predator fishery resource model with an impulsive supply of fish breed and harvesting. Using the inequalities of impulsive differential equations, the system is proved to be bounded. Then the conditions for stability of system are established using Floquet theory and comparison principle. Finally, numerical simulation is performed to validate the theoretical results.

Keywords: Prey, Predator, Harvesting, Impulsive Differential Equation, Local Stability, Global Stability.

I. Introduction

The topic of harvesting in marine biology is of great interest to economists and ecologists. Marine biology is a great source of food, medicine and raw materials. However, unregulated and extensive harvesting of fish species may lead to the extinction of some fish species. During the last two decades, a number of researchers have studied the models of fishery resource [2-7, 11].

Lv et. al. [1] proposed a phytoplankton-zooplankton model and studied the effect of overexploitation of the species. In [2], the authors proposed a prey-predator fishery model in which both the species are harvested and studied the effect of toxicants released by some other species.

Dubey [3] proposed a prey-predator mathematical model with a reserve area (where fishing is prohibited) and studied the role of reserve area in the dynamics of the system. Kar [4] also considered a fishery model with reserve area and proved that under certain conditions, the system is locally as well as globally stable. Here Pontryagin's Maximal Principle is used to discuss the optimal harvesting policy.

Sharma and Gupta [5] studied the dynamics of fishery resource in with reserve area in the presence of bird predator. The authors of [6] studied the effect of density of fishes on the dynamics of the fishery resource model with reserve area by considering the modified effort function as suggested by Idels and Wang [7]. Dhar and Jatav [8] considered a delayed stage-structured predator-prey model with the impulsive diffusion of prey between two patches of predator populations and obtained a condition of global attractivity of the predator-free periodic solution. In [11], the authors proposed a Holling type-II functional response fishery model and suggested that until the reserved zone does not extinct, the system will remain sustainable.

In all the papers discussed above it was assumed that the harvesting is a continuous process but in real situations the due to several socio-economic constraints and seasonal factors, the harvesting/diffusion between two patches may be impulsive. Some authors have recently studied the impulsive fishery resource models. Wang and Jia [13] proposed and studied a single species model with impulsive diffusion and pulse harvesting at different fixed times.

The purpose of this article is to study harvesting of a prey-predator fishery resource two patch system: one reserved and other unreserved. Each patch is supposed to be homogeneous. In reserve area no fishing and predation is permitted, whereas in the unreserved patch, fishing as well as predation is allowed. The growth of prey in the unreserved patch in the absence of predator is assumed to be logistic and in reserve area fish breed is added impulsively. The predator consumes the prey in the unreserved area and fishing is performed impulsively.

The organization of the paper is as follow. In section 2, we develop a prey-predator fish harvesting model via impulsive differential equations. In section 3, it is proved that the system under consideration is bounded. Using Floquet's theory, small amplitude perturbation technique and comparison principles, sufficient condition for local and global stability of predator-free periodic solution is obtained in section 4. Numerical simulation and discussions are done in section 5.

II. Mathematical Model

Before modeling the system in mathematical terms, we have the following assumptions:

A_1: The total population of fish species is divided into two patches: reserved and unreserved area.

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A_2: The fish population moves freely from the reserve area to the unreserved area.

A_3: The predator fish attack fish in the unreserved area only with Holling type II functional response.

A_4: At time $t = nT,\ n \in Z_+ = \{1, 2, 3, \dots\}$ the fish population and predator in the unreserved area are harvested periodically.

A_5: The fish breed is added only in the reserved area.

With these assumptions, consider the following fish harvesting prey-predator model

$$\begin{cases} \left.\begin{aligned} \frac{dx(t)}{dt} &= rx(t)\left(1-\frac{x(t)}{K}\right)+\sigma Y(t)-\frac{\mu x(t)z(t)}{\alpha+x(t)} \\ \frac{dy(t)}{dt} &= -\sigma Y(t) \\ \frac{dz(t)}{dt} &= \frac{\beta x(t)z(t)}{\alpha+x(t)}-dz(t) \end{aligned}\right\} & t \neq nT \\ \left.\begin{aligned} x(t^+) &= (1-q_1E)x(t) \\ y(t^+) &= y(t)+\theta \\ z(t^+) &= (1-q_2E)z(t) \end{aligned}\right\} & t = nT, n = 1,2,3 \dots\dots \end{cases} \quad (1)$$

where

x(t): Fish population in the unreserved area.

y(t): Fish population in reserve area.

z(t): Fish predator population in the unreserved area.

r: The intrinsic growth rate of the fish population in the unreserved area.

K: Carrying capacity of the unreserved area.

σ: Amount of fish population migrated from reserved to the unreserved area.

μ: The rate of predation done by a predator in the unreserved area.

β: The rate of conversion of predation into the growth of predator.

d: The natural death rate of a predator.

α: The handling time spent by a predator on processing a fish.

E: Effort applied for harvesting.

q_1: Catchability coefficient for fish population in the unreserved area.

q_2: Catchability coefficient for predator fish population in the unreserved area.

θ: Amount of fish breed added in reserve area.

III. BOUNDEDNESS

Here we prove the boundedness of the prey-predator fishery resource model (1).

Theorem 3.1 There exists a positive constant L such that $x(t) \le L,\ y(t) \le L,\ z(t) \le L$, for each solution of (1) with t being large enough.

Proof: Define $v(t) = x(t) + y^2(t) + \frac{\mu}{\beta} z(t)$ and let $0 < \gamma < d$.

Then for $t \neq nT$, we obtain that

$$D^+v(t) + \gamma v(t) \le \frac{(r+\gamma)^2}{4r} + \frac{\sigma^2}{4(2\sigma-\gamma)} = L_0.$$

When $= nT,\ v(t^+) \le v(t) + \theta$.

By Lemma 3.1 of [10] for $t \in (nT, (n+1)T]$, we get

$$v(t) \le v(0)\exp(-dt) + \int_0^t L_0 \exp\big(-\gamma(t-s)\big)\, ds$$

$$+\textstyle\sum_{0<nT<t} \theta \exp(-\gamma(t-nT))$$

$$\to \frac{L_0}{d} + \frac{\theta \exp(-\gamma T)}{\exp(\gamma t) - 1}, \text{ as } t \to \infty,$$

which implies that $v(t)$ is uniformly bounded. Therefore, by definition of $v(t)$, there exists a constant

$$L := \frac{L_0}{d} + \frac{\theta\exp(-\gamma T)}{\exp(\gamma t) - 1}$$

such that $S_1(t) \le L, S_2(t) \le L, E(t) \le L, I(t) \le L$ and $N(t) \le L$ for all t large enough.

IV. STABILITY

In this section, we shall discuss the local stability of various equilibrium points of the system (1) using tools impulsive differential equations. Then the sufficient condition for predator-free periodic solution is also established.

When $z(t) = 0$ for all $t \ge 0$, we rewrite system (1) as below:

$$\begin{cases} \left.\begin{aligned} x'(t) &= rx(t)\left(1-\frac{x(t)}{K}\right)+\sigma Y(t) \\ y'(t) &= -\sigma Y(t) \end{aligned}\right\} & t \neq nT \\ \left.\begin{aligned} x(t^+) &= (1-q_1E)x(t) \\ y(t^+) &= y(t)+\theta \end{aligned}\right\} & t = nT \end{cases} \quad (2)$$

By Lemma 2.1 of [9] and Lemma 3.3 of [10], we get the following result

$$x^*(t) = \frac{1}{1-(1-q_1E)e^{rT}}\left(\frac{\sigma\theta}{\sigma+r}\frac{e^{r(t-nT)}\left(1-e^{-(\sigma+r)(t-nT)}\right)}{1-e^{-\sigma T}}\right)$$

$$x^*(0^+) = \frac{1-q_1E}{1-(1-q_1E)e^{rT}}\left(\frac{-\sigma\theta}{\sigma+r}\frac{e^{\sigma T}-e^{rT}}{1-e^{-\sigma T}}\right)$$

$$y^*(t) = \frac{\theta e^{-\sigma(t-nT)}}{1-e^{-\sigma T}},$$

$$y^*(0^+) = \frac{\theta}{1-e^{-\sigma T}}.$$

Theorem 4.1

Let$\big(x(t),\ y(t),\ z(t)\big)$ be any solution of (1), then

(i)The trivial solution$(0, 0, 0)$of sythe stem (1) is locally asymptotically stable provided:

$$(1-q_1E)e^{rT} < 1.$$

(ii)The predator free periodic solution $(x^*(t), y^*(t), 0)$ is locally asymptotically stable provided:

$$(1-q_1E)e^{\int_0^T\left(r-\frac{2r}{K}x^*(t)\right)dt} < 1 \quad \text{and}$$

$$(1-q_1E)e^{\int_0^T\left(\frac{\beta x^*(t)}{\alpha+x^*(t)}-d\right)dt} < 1.$$

Proof:(i) To prove that a trivial solution is locally asymptotically stable, we use small-amplitude perturbation method. Let

$$x(t) = \emptyset_1(t);\ y(t) = \emptyset_2(t)\ ;\ z(t) = \emptyset_3(t)$$

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where $\emptyset_1(t), \emptyset_2(t)$ and $\emptyset_3(t)$ are small perturbations.
The system (1) can be rewritten as:

$$\begin{cases} \left.\begin{array}{l} \frac{d\emptyset_1(t)}{dt} = r\emptyset_1(t) + \sigma\emptyset_2(t) \\ \frac{d\emptyset_2(t)}{dt} = -\sigma\emptyset_2(t) \\ \frac{d\emptyset_3(t)}{dt} = -d\emptyset_3(t) \end{array}\right\} & t \neq nT \\ \left.\begin{array}{l} \emptyset_1(t^+) = (1-q_1E)\emptyset_1(t) \\ \emptyset_2(t^+) = \emptyset_2(t) + \theta \\ \emptyset_3(t^+) = (1-q_2E)\emptyset_3(t) \end{array}\right\} & t = nT, n = 1,2,3, \dots \end{cases} \quad (3)$$

Let $\Phi(t)$ be the fundamental matrix of (3), it must satisfy

$$\frac{d\Phi(t)}{dt} = \begin{bmatrix} r & \sigma & 0 \\ 0 & -\sigma & 0 \\ 0 & 0 & -d \end{bmatrix} \Phi(t) = A\Phi(t)$$

The linearization of impulsive conditions

$$\begin{bmatrix} \emptyset_1(t^+) \\ \emptyset_2(t^+) \\ \emptyset_3(t^+) \end{bmatrix} = \begin{bmatrix} 1-q_1E & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1-q_2E \end{bmatrix} \begin{bmatrix} \emptyset_1(t) \\ \emptyset_2(t) \\ \emptyset_3(t) \end{bmatrix}$$

Thus the monodromy matrix is

$$M = \begin{bmatrix} 1-q_1E & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1-q_2E \end{bmatrix} \Phi(t)$$

$\Phi(t) = \Phi(0)exp\left(\int_0^T A\, dt\right)$, where $\Phi(0)$ is identity matrix. Then the eigen values of the monodromy matrix M are

$$\lambda_1 = (1-q_1E)e^{rT},$$
$$\lambda_2 = e^{-\sigma T} < 1,$$
$$\lambda_3 = (1-q_1E)e^{-dT} < 1.$$

The trivial solution $(0, 0, 0)$ of system (1) is locally asymptotically stable provided $\lambda_1 < 1$.

(ii) Now for local stability of predator free periodic solution $(x^*(t), y^*(t), 0)$, we again use the same method as in case (i).

$x(t) = \emptyset_1(t) + x^*(t);\ y(t) = \emptyset_2(t) + y^*(t)$;
$z(t) = \emptyset_3(t)$

where $\emptyset_1(t), \emptyset_2(t)$ and $\emptyset_3(t)$ are small perturbations. In a similar manner as in above case, we obtain:

In a similar manner as in above case, we obtain:

$$\begin{cases} \frac{d\emptyset_1(t)}{dt} = r\emptyset_1(t) - \frac{2r}{K}x^*(t) + \sigma\emptyset_2(t) - \frac{\mu x^*(t)}{\alpha + x^*(t)}\emptyset_3(t) & \\ \frac{d\emptyset_2(t)}{dt} = -\sigma\emptyset_2(t) & t \neq nT \\ \frac{d\emptyset_3(t)}{dt} = \frac{\beta x^*(t)}{\alpha + x^*(t)}\emptyset_3(t) - d\emptyset_3(t) & \\ \emptyset_1(t^+) = (1-q_1E)\emptyset_1(t) & \\ \emptyset_2(t^+) = \emptyset_2(t) + \theta & t = nT,\ n = 1,2,3, \dots \\ \emptyset_3(t^+) = (1-q_2E)\emptyset_3(t) & \end{cases} \quad (4)$$

Let $\Phi(t)$ be the fundamental matrix of (4), it must satisfy

$$\frac{d\Phi(t)}{dt} = \begin{bmatrix} r - \frac{2r}{K}x^*(t) & \sigma & -\frac{\mu x^*(t)}{\alpha + x^*(t)} \\ 0 & -\sigma & 0 \\ 0 & 0 & \frac{\beta x^*(t)}{\alpha + x^*(t)} - d \end{bmatrix} \Phi(t)$$
$$= A\Phi(t)$$

The linearization of impulsive conditions

$$\begin{bmatrix} \emptyset_1(t^+) \\ \emptyset_2(t^+) \\ \emptyset_3(t^+) \end{bmatrix} = \begin{bmatrix} 1-q_1E & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1-q_2E \end{bmatrix} \begin{bmatrix} \emptyset_1(t) \\ \emptyset_2(t) \\ \emptyset_3(t) \end{bmatrix}$$

Thus the monodromy matrix is

$$M = \begin{bmatrix} 1-q_1E & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1-q_2E \end{bmatrix} \Phi(t)$$

$\Phi(t) = \Phi(0)exp\left(\int_0^T A\, dt\right)$, where $\Phi(0)$ is identity matrix. Then the eigen values of the monodromy matrix M are

$$\lambda_1 = (1-q_1E)e^{\int_0^T \left(r - \frac{2r}{K}x^*(t)\right) dt},$$

$$\lambda_2 = e^{-\sigma T} < 1,$$

$$\lambda_3 = (1-q_1E)e^{\int_0^T \left(\frac{\beta x^*(t)}{\alpha + x^*(t)} - d\right) dt}.$$

The predator-free periodic solution$(x^*(t), y^*(t), 0)$ of system (1) is locally asymptotically stable provided $\lambda_1 < 1$ and $\lambda_3 < 1$ and hence the result.

Remark: The solution $\left(0, 0, z(t)\right), z(t) > 0$ does not exist for system (1). This type of solution is possible only when supply of predator in considered.

Theorem 4.2 The periodic solution $(x^*(t), y^*(t), 0)$ is globally asymtotically stable for the system (1) provided

$$\int_{n_2T}^{t} \left[\frac{\beta(x^*(t) + \epsilon_1)}{\alpha + x^*(t) + \epsilon_1} - d\right] dt < 0.$$

Proof: We have already proved that $(x^*(t), y^*(t), 0)$ is locally asymptotically stable. Therefore, we only need to prove it is globally attractive.
we can choose a ϵ_1 small enough such that

$$\int_{n_2T}^{t} \left[\frac{\beta(x^*(t) + \epsilon_1)}{\alpha + x^*(t) + \epsilon_1} - d\right] dt = \sigma < 0.$$

Besides, We have

$$x'(t) = rx(t)\left(1 - \frac{x(t)}{K}\right) + \sigma Y(t) \le rx(t) + \sigma Y(t).$$

From Lemma 3.3 of [6], there exists a n_1 such that for $x(t) \le x^*(t) + \epsilon_1$, for $t \ge n_1T$,
Similarly, there exists a n_2 $(n_2 > n_1)$ such that $y(t) \le y^*(t) + \epsilon_1$, for $t \ge n_2T$.
Thus, for $t \ge n_2T$, we have

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$$z'(t) = \left(\frac{\beta x(t)}{\alpha + x(t)} - d\right)z(t) \leq \left(\frac{\beta(x^*(t) + \epsilon_1)}{\alpha + x^*(t) + \epsilon_1} - d\right)z(t)$$

From the above inequality, we get

$$z(t) \leq z(n_2 T)e^{\int_{n_2 T}^{t}\left[\frac{\beta(x^*(t)+\epsilon_1)}{\alpha + x^*(t)+\epsilon_1} - d\right]dt}$$

$$\leq z(n_2 T)e^{k\sigma},$$

where $t \in ((n_2 + k)T, (n_2 + k + 1)T], k \in Z_+$.
Since $\sigma < 0$, we can easily see that $z(t) \to 0$ as $k \to +\infty$. Thus for arbitrary positive constant ϵ_2 small enough, there exist $n_3(n_3 > n_2)$ such that
$z(t) < \epsilon_2$ for all $t \geq n_3 T$. From which we get

$$x'(t) = rx(t)\left(1 - \frac{x(t)}{K}\right) + \sigma Y(t) - \frac{\mu x(t)z(t)}{\alpha + x(t)}$$
$$\geq \left(rx(t)\left(1 - \frac{x(t)}{K}\right) + \sigma Y(t) - \frac{\mu x(t)\epsilon_2}{\alpha + x(t)}\right),$$

From Lemma 3.3 of [6], there exists a $n_4\,(n_4 > n_3)$ such that $x(t) \geq x_2^*(t) - \epsilon_1$, for $t \geq n_4 T$,

By similarly argument, there exists a $n_5\,(n_5 > n_4)$ such that $y(t) \geq y_2^*(t) - \epsilon_1$, for $t \geq n_5 T$,

Note that ϵ_1, ϵ_2 are positive constants small enough and $x_2^*(t) \to x^*(t)$, $y_2^*(t) \to y^*(t)$ as $t \to \infty$. Therefore, the periodic solution $(x^*(t), y^*(t), 0)$ is globally asymtotically stable.

V. Numerical Simulation and Discussion

In this paper, we study harvesting of prey-predator model with Holling type-II functional response in a two patch environment. The purpose of this section is to numerically investigate the effect of impulsive harvesting of fish and fish predator population. For this purpose, we performed numerical simulation of system (1) with the values of parameters given in table 1. The values of parameters are taken per week as $x(0^+)$ =0.5, $y(0^+)$ =0.5 and $z(0^+)$ = 0.5.

We have proved in Theorem 4.1 and Theorem 4.2 that predator free periodic solution is locally as well as globally stable and for the given set of parameters Fig. 1 shows predator free periodic solution $(x^*(t), y^*(t), 0)$ is globally asymptotically stable. Further, the numerical simulation was performed to verify the co-existence of all populations. Fig 2 shows that there exists a stage when all the populations $x(t), y(t)$ and $z(t)$ co-exist and the system is said to be permanent. For the parametric values for which system is permanent, in Fig 3 bifurcation plot shows the complex dynamical behavior of all the populations.

VI. CONCLUSION

Study of harvesting and preservation of fishery resources is a need of time. In this paper, an impulsively harvesting prey-predator model with reserve area is proposed and analyzed where both the prey and predator are fish species. It was also assumed that fish breed is also added impulsively in reserve area and harvesting is allowed in unreserved area only. The system is proved to be bounded and sufficient conditions are established for local stability of predator free periodic solution $(x^*(t), y^*(t), 0)$. The predator free periodic solution is proved to be globally asymptotically stable. Then choosing certain parametric values, the system is investigated numerically. Fig 1 justifies the results proved theoretically. Fig 2 shows that for certain vales of parameters the system may be permanent and Fig 3 shows that the interaction between the species forms a complex dynamical behavior.

Table 1. Parametric values chosen for simulation

Para--meter	**Description**	**Value per week**
r	Growth rates of prey fish species within unreserved area	6
K	Carrying capacity of the prey fish species inside the unreserved area.	5
σ	Migration rate from reserved to unreserved area.	2
μ	The maximum uptake rate.	1
α	The half-saturation constant.	1
β	The ratio biomass conversion.	0.8
d	Natural death rate of predator species.	0.3
q_1	Catchability coefficients of prey fish species inside unreserved area	0.1
q_2	Catchability coefficients of predator fish species inside unreserved area	0.2
θ	Amount of fish breed supplied impulsively.	0.5
E	Effort applied to harvest prey and predator fish.	2

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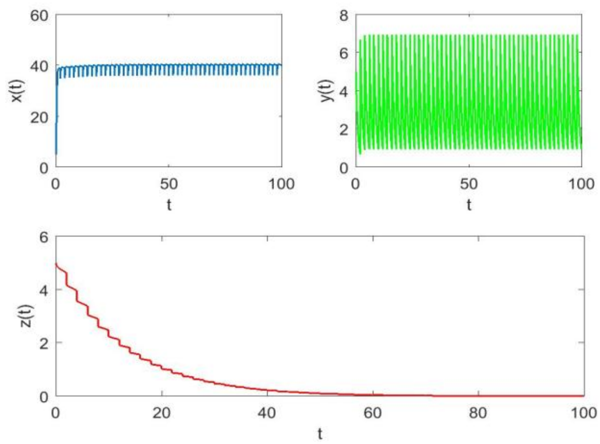

Fig 1 Time-series plot of $x(t), y(t)$ and $z(t)$ shows that predator free periodic solution is globally stable.

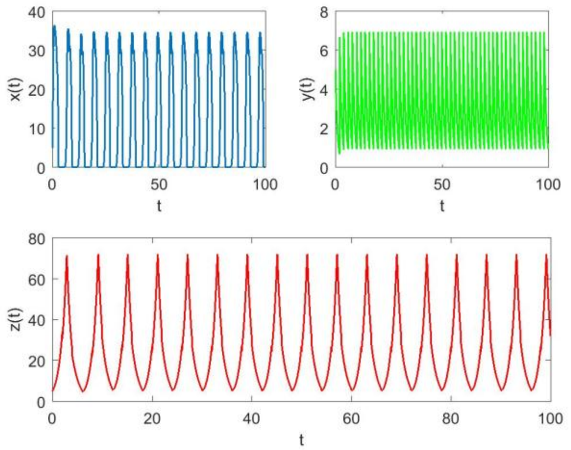

Fig 2 Time-series plot of $x(t), y(t)$ and $z(t)$ shows that periodic solution $\left(x^*(t), y^*(t), z^*(t)\right)$ is permanent.

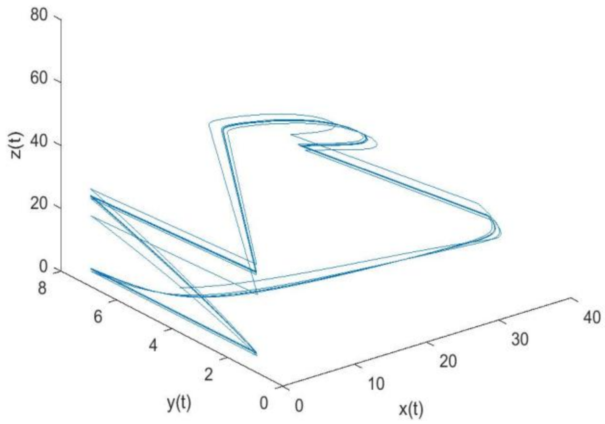

Fig 3 Bifurcation plot shows the complex dynamical behavior of all populations.

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