On The Energy of Signed Graphs

Gurdip Singh Sodi

Department of Computer Science, SAM Govt. Degree College Budgam, J&K, India.

Abstract: Let S be a signed graph of order n and let $\mu_1, \mu_2, \ldots, \mu_n$ bet its adjacency eigenvalues. Then the energy of S is defined as $E(S) = \sum_{j=1}^{n} |\mu_j|$. In this paper, we characterize positive rational numbers which can be the energy of a signed graph and obtain related results.

Keywords – Signed Graph, Rational Number, eigenvalues.

I. Introduction

A signed graph is defined to be a pair $S = (G, \sigma)$, where $G = (V, E)$ is the underlying graph and $\sigma: E \rightarrow \{-1,1\}$ is the sign function or signature. Our signed graphs will have simple underlying graphs. A signed graph is said to be homogenous if all of its edges have same sign and heterogeneous, otherwise. A graph G can be considered as all-positive signed graph and in this sense signed graphs are generalization of graphs. The sign of a signed graph is defined as the product of signs of its edges. The sign of a signed subgraph is the restriction of sign function to the subgraph. A cycle in a signed graph is positive if its sign is positive i.e., it contains an even number of negative edges. A signed graph is balanced if each of its cycle is positive and unbalanced, otherwise. For undefined notations and terminology see [7].

The adjacency matrix of a signed graph $S = (G, \sigma)$ with vertex set $v_1, v_2, \ldots, v_n$ is an $n \times n$ matrix $A(S) = (a_{ij})$, where $a_{ij} = \sigma(v_i v_j)$, if there is an edge from vertex v_i to v_j and zero, otherwise. The eigenvalues of $A(S)$ are called the eigenvalues of S. The eigenvalues of a signed graph S is also called spectrum of S and we denote this by $spec(S) = \{\mu_i; i = 1,2, \ldots, n\}$.

Gutman [11] defined the energy of a graph as the sum of absolute values of its eigenvalues . Germina , Hameed and Zaslavsky [10] introduced the concept of energy in signed graph and defined the energy of a signed graph as the sum of absolute values of its eigenvalues i.e., if $\mu_1, \mu_2, \ldots, \mu_n$ are eigenvalues of S then the energy of S is defined as $E(S) = \sum_{j=1}^{n} |\mu_j|$.

We recall the definitions of Cartesian product and conjunction of two signed graphs from [7].

Let $S_1(V_1, E_1, \sigma_1)$ and $S_2(V_2, E_2, \sigma_2)$ be two signed graphs with respective orders n_1 and n_2 . The ***Cartesian product*** of S_1 and S_2 denoted by $S_1 \times S_2$ is a signed graph whose vertex set is $V = V_1 \times V_2$. Two vertices (x, y) and (p, q) are adjacent if either there is an edge xp in S_1 or yq is edge in S_2. Moreover, sign of edge $(x,y)(p,q)$ is $\sigma_1(xp)$ or $\sigma_2(yq)$ according as xp is an edge in S_1 or yq is edge in S_2.

Conjunction of S_1 and S_2 denoted by $S_1 * S_2$ is a signed graph whose vertex set is $V = V_1 \times V_2$. Two vertices (x, y) and (p, q) are adjacent if there is an edge xp in S_1 and yq is edge in S_2. Moreover, sign of edge $(x,y)(p,q)$ is $\sigma_1(xp)\, \sigma_2(yq)$.

Lemma 1 [7]. Let S_1 and S_2 be two signed graphs with respective order n_1, n_2 and let their spectra be $spec(S_1) = \{\theta_i; i = 1,2, .., n_1\}$ and $spec(S_2) = \{\delta_j; j = 1,2, .., n_2\}$. Then the spectrum of Cartesian product and the conjunction of S_1 and S_2 is given by $spec\,(S_1 \times S_2) = \{\theta_i + \delta_j; i = 1,2, .., n_1 \,\&\, j = 1,2, .., n_2\}$ and $spec\,(S_1 * S_2) = \{\theta_i \delta_j; i = 1,2, .., n_1 \,\&\, j = 1,2, .., n_2\}$.

From [8], we see that any rational root of monic polynomial having integral coefficients is an integer or equivalently any rational algebraic integer is an integer. Since characteristic polynomial of a signed graph is a monic polynomial with integral coefficients, therefore we have the following observation.

Lemma 2. If the eigen value of a signed graph is rational, then it is necessarily an integer.

Main Results.

Bapat and Pati [2] characterized positive rational numbers which be the energy of a graph. They proved that energy of a graph whenever rational is necessarily an even integer. We prove a similar result for signed graphs.

Theorem 3. Energy of a signed graph whenever rational is necessarily an even integer.

Proof. Let S be a signed graph of order n and let $\mu_1, \mu_2, \ldots, \mu_n$ be its eigenvalues. Then the energy of S is given as

$$E(S) = \sum_{j=1}^{n} |\mu_j|.$$

Since S is simple signed graph, therefore trace of adjacency matrix of S is zero. But trace of a square matrix is sum of its eigenvalues, therefore we have

$$\sum_{j=1}^{n} \mu_j = 0.$$

This gives that

$$\sum_{+} \mu_j = \sum_{-} \mu_j,$$

 DOI : 10.18231/2454-9150.2018.0602

ISSN : 2454-9150 Vol-04, Issue-05, Aug 2018

where $\sum_{+} \mu_j$ and $\sum_{-} \mu_j$ respectively denote sum of positive and sum of negative eigenvalues. Therefore the energy of S is given by $E(S) = 2\sum_{+} \mu_j$

Let $\mu_{1,} \mu_{2,} \dots, \mu_m$ be positive eigenvalues of S. Then the energy of S is

$$E(S) = 2\sum_{i=1}^{m} \mu_i \qquad (1)$$

By using Lemma 1, we see that $\mu = \sum_{i=1}^{m} \mu_i$ is an eigenvalue of the Cartesian product of m copies of S i.e., $S \times S \times \dots \times S\ (m\ times)$. Now, μ being sum of algebraic integers is also an algebraic integer. Also, from (1) together with the fact that a rational algebraic integer is an integer, we see that energy of signed graph whenever rational is necessarily an even number.

Theorem 4. Energy of a signed graph cannot be rth root of a positive non-integral rational number.

Proof. Let S be a signed graph of order n. Then by similar theory as in Theorem 3, we see that $E(S) = 2\mu$.

Assume on contrary that $(S) = (\frac{p}{q})^{\frac{1}{r}}$, where $\frac{p}{q}$ is a positive non-integral rational number in lowest form and r is a positive integer.

This implies $2\mu = (\frac{p}{q})^{\frac{1}{r}}$, which gives $\mu^r = \frac{p}{2^r q}$.

As $\frac{p}{q}$ is positive non-integral rational number, therefore $\frac{p}{2^r q}$ is also non-integral positive rational number and hence μ^r is non-integral positive rational. But μ^r is an eigenvalue of a signed graph obtained by conjunction of r copies of S i.e., $S * S * \dots * S\ (r\ times)$. Now μ^r being an eigenvalue of a signed graph is an algebraic integer and hence cannot be non-integral rational, which gives a contradiction.

Pirzada and Gutman [12] proved that the energy of a signed graph cannot be the square root of an odd integer. A similar result holds for signed graphs.

Theorem 5. Energy of signed graph cannot be of the form $(2^s q)^{\frac{1}{r}}$, where q is odd, $r \geq 1$ and $0 \leq s \leq r-1$.

Proof. Let S be a signed graph of order n and let $\mu_{1,} \mu_{2,} \dots, \mu_m$ be its positive eigenvalues. Then as in Theorem 3, the energy of S is given by $E(S) = 2\sum_{i=1}^{m} \mu_i$

Now, let $\mu = \sum_{i=1}^{m} \mu_i$. Then $E(S) = 2\mu$.

Let H denote the signed graph obtained by taking Cartesian product of m copies of S and K denote the signed graph obtained by taking conjunction of r copies of H i.e., $H = S \times S \times \dots \times S\ (m\ times)$ and $K = H * H * \dots * H\ (r\ times)$. By Lemma 1, μ is an eigenvalue of H and μ^r is an eigenvalue of K.

If possible, let $E(S) = (2^s q)^{\frac{1}{r}}$. Then $2\mu = (2^s q)^{\frac{1}{r}}$, which gives $= \frac{2^s q}{2^r}$

As q is odd, it is clear that if $0 \leq s \leq r-1$, then μ^r is a non-integral positive rational eigenvalue of

the signed graph K, which is a contradiction.

Remark 6 . In Theorem 5, if $s = 0$ and $r = 1$, then we see that the energy of a signed graph cannot be square root of an odd integer.

REFERENCES.

[1]. B. D. Acharya, Spectral criterion for the cycle balance in networks. J. Graph Theory 4(1980)1-11.

[2]. R. B. Bapat and S. Pati, Energy of a graph is never an odd integer, Bull. Kerala Math. Assoc., 1(2004) 129-132.

[3] M. A. Bhat and J. Rada, Lower bounds for the energy of weighted digraphs, Linear Multilinear Algebra (2018), to appear

[4]M. A. Bhat, Energy of weighted digraphs, Discrete Appl. Math. 223 (2017) 1-14.

[5] M. A. Bhat and S. Pirzada, Unicyclic signed graphs with minimal energy, Discrete Appl. Math 226 (2017) 32-39.

[6] M. A. Bhat and S. Pirzada, Spectra and energy of bipartite signed digraphs, Linear Multilinear Algebra, 64, 9 (2016) 1863-1877.

[7] M. A. Bhat and S. Pirzada, On equienergetic signed graphs, Discrete Appl. Math 189 (2015)1-7.

[8]. Birkhoff and Mac Lane: A survey of Modern Algebra, Universities Press (India) private limited 2003.

[9]. K. A. Germina and Shahul Hameed K, On signed paths, signed cycles and their energies. Appl. Math. Sci. 4(2010), no. 70, 3455-3466.

[10]. K. A. Germina, Shahul Hameed K and Thomas Zaslavsky, On products and line graphs of signed graphs , their eigenvalues and energy, Linear Algebra and its Applications, 435, 10 (2010) 2432-2450.

[11]. I. Gutman, The Energy of a graph, Ber. Math. Statist. Sekt. Forschungszenturm Graz., 103(1978) 1-22.

[12]. S. Pirzada and I. Gutman, Energy of a graph is never the square root of an odd integer, Appl. Anal. Disc. Math 2 (2008) 118-121.

[13]. S. Pirzada and M. A. Bhat, Energy of signed digraphs, Discrete Appl. Math. 169 (2014) 195-205.

[14]. S. Pirzada , An introduction to graph theory, Universities Press, Orient BlackSwan, Hyderabad, India, 2012.

 DOI : 10.18231/2454-9150.2018.0602