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Line Subdivision Double Domination in Graphs

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ABSTRACT - Let $S(G)$ be the subdivision graph of G. The line graph of of $S(G)$, $L(S(G))$ is a graph whose vertices correspond to the edges of $S(G)$ and two vertices in $L[S(G)]$ are adjacent if and only if corresponding edges in $S(G)$ are adjacent. A subset D^d of $V[L(S(G))]$ is double domination set of $L[S(G)]$ if for every vertex $v \in V[L(S(G))]$, $|N[v] \cap D^d| \geq 2$, that is v is in D^d and has at least one neighbour in D^d or v is in $V[L(S(G))] - D^d$ and has at least two neighbours in D^d. The line subdivision double domination number $\gamma_{ddls}(G)$ is a minimum cardinality of the line subdivision double dominating set of G and is denoted by $\gamma_{ddls}(G)$. In this paper, we establish some upper and lower bounds on $\gamma_{ddls}(G)$ in terms of the vertices, edges and other different parameters of G and not in terms of the elements of $L[S(G)]$. Further, its relation with other different dominating parameters is also obtained. The main deal of this paper is to apply a probabilistic approach to obtain new bounds for line subdivision double domination parameter and to study their relationship with other different domination parameters of different graph valued functions.

SUBJECT CLASSIFICATION NUMBER: $AMS - 05C69, 05C70$.

KEYWORD: ***Line Subdivision Graph/Dominating set/Double domination***.

I. INTODUCTION

In this paper, all the graphs considered here are simple, finite, non-trivial, undirected and connected. The vertex set and edge set of graph G are denoted by $V(G) = p$ and $E(G) = q$ respectively. The terms not defined here are used in the sense of Harary [9]. The neighbourhood of a vertex $v \in V$ is defined by $N(v) = \{u \in V / uv \in E\}$. The close neighbourhood of a vertex$v$ is $N[v] = N(v) \cup \{v\}$. The order $|V(G)|$ of G is denoted by p. A vertex cover in a graph G is a set of vertices that covers all the edges of G. The vertex covering number $\alpha_0(G)$ is the minimum cardinality of a vertex cover in G. A set of vertices in a graph G is called independent set if no two vertices in the set are adjacent. The vertex independence number $\beta_0(G)$ is the maximum cardinality of an independent set of vertices. A set D of vertices in a graph G is called a dominating set of G if every vertex in $V - D$ is adjacent to some vertex in D. The domination number of G, denoted by $\gamma(G)$ is the minimum cardinality of a dominating set.The domination in graphs with many variations is now well studied in graph theory. A thorough study of domination appears in [14]. Let $S(G)$ be the subdivision graph of G. The line graph of $S(G)$ is a graph whose vertices correspond to the edges of $S(G)$ and two vertices in $L[S(G)]$ are adjacent if and only if corresponding edges in $S(G)$ are adjacent. A subset D^d of $V[L(S(G))]$ is double dominating set of $L[S(G)]$ if for every vertex $v \in V[L(S(G))]$, $|N[v] \cap D^d| \geq 2$, that is v is in D^d and has at least one neighbour in D^d or v is in $V[L(S(G))] - D^d$ and has at least two neighbours in D^d. The line subdivision double dominating number $\gamma_{ddls}(G)$ is a minimum cardinality of the line subdivision double dominating set of G and is denoted by $\gamma_{ddls}(G)$. The graphvalued function related to double domination parameters have been studied in [1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, and 16]. Further in [15], studied the domination subdivision number. In this paper, we establish some upper and lower bounds on $\gamma_{ddls}(G)$ in terms of the vertices, edges and other different parameters of G and not in terms of the element of $L[S(G)]$. Further, its relation with other different dominating parameters is also obtained.

Observation 1: Let G be a graph that admits line subdivision double domination set D^d. Then $|D^d| \geq p + \delta(G)$ if and only if $G = P_n$ with $p \geq 3$.

Observation 2: For any connected (p, q) graph G, $p - \gamma_{ddls}(G) \leq 0$.

Observation 3: For any connected (p, q) graph G, $\gamma_{dd}(G) \leq \gamma_{ddls}(G)$.

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II. LOWER BOUNDS FOR $\gamma_{ddls}(G)$.

We establish lower bound for $\gamma_{ddls}(G)$ in terms of elements of G.

Theorem 2.1: For any connected (p,q) graph G with $p \geq 4, \gamma_{ddls}(G) \geq \gamma_{ct}[L(G)] + \gamma_{ct}(G)$.

Proof: Let $A = \{ v_1, v_2, \dots, v_n \} \subseteq V(G)$ be the minimal set of vertices which covers all the vertices in G such that $|S| = \gamma_{ct}(G)$. Now in $L(G)$, let $D = \{u_1, u_2, \dots, u_k\} \subseteq V[L(G)]$ be the minimum set of vertices such that for every $x \in N(u)$ where $x \in V[L(G)] - D, u \in D$ and $N(x) \neq \emptyset$ in $V[L(G)] - D$. Clearly D forms a minimal cototal dominating set of $L(G)$. Let $B = \{ v_1, v_2, \dots, v_j \} \subseteq V[S(G)]$ be the set of all non end vertices in $S(G)$ and let $C = \{ e_1, e_2, \dots, e_k \}$ be the set of edges which are incident to the vertices of B. Now in L[S(G)], suppose $I = \{u_1, u_2, \dots, u_i\} \subseteq V[L(S(G))]$ be the set of vertices with $\deg(u_i) = 1, 1 \leq i \leq n$. Then $D^d = I \cup C'$ where $C' = \{u_1, u_2, \dots, u_m\} \subseteq C$ in $L[S(G)]$ corresponding to the edges of C from a double dominating set of $L[S(G)]$. Therefore it follows that $|D^d| \leq |D| \cup |S|$. Hence $\gamma_{ddls}(G) \leq \gamma_{ct}[L(G)] + \gamma_{ct}(G)$.

Theorem 2.2: For any connected (p,q) graph G, $\gamma_{ct}[L(G)] + diam(G) \leq \gamma_{ddls}(G)$.

Proof: Let $E' = \{e_1, e_2, \dots, e_n \} \subseteq E(G)$ be the minimal set of edges which constitute the longest path between any two distinct vertices $u, v \in V(G)$, such that $d(u,v) = diam(G)$. Now let $A = \{ v_1, v_2, \dots, v_n \} \subseteq V(G)$ be the set of all non end vertices in G. Further, let $F = \{ e_1, e_2, \dots, e_k \}$ be the set of edges which are incident to the vertices of A. Now in L(G), suppose $I = \{u_1, u_2, \dots, u_i\} \subseteq V[L(G)]$ be the set of vertices with $\deg(u_i) = 1, 1 \leq i \leq n$. Then $D = I \cup F'$, where $F' = \{ u_1, u_2, \dots, u_i \} \subseteq F$ in $L(G)$ corresponding to the edges of F forms a cototal dominating set of $L(G)$. Since $V[L(S(G))] = E[S(G)]$, let $B = \{ v_1, v_2, \dots, v_j \} \subseteq V[S(G)]$ be the set of all nonend vertices in $S(G)$ and let $C = \{e_1, e_2, \dots, e_k \}$ be the set of edges, which are incident to the vertices of B. Now in $L[S(G)]$, suppose $I = \{u_1, u_2, \dots, u_i\} \subseteq V[L(S(G))]$ be the set of vertices with $\deg(u_i) = 1, 1 \leq i \leq n$. Then $D^d = I \cup C'$ where $C' = \{ u_1, u_2, \dots, u_i \} \subseteq C$ in $L[S(G)]$ corresponding to the edges of C form a double dominating set of $L[S(G)]$. It follows that $|D| \cup dist(u,v) \leq |D^d|$ and hence $\gamma_{ct}[L(G)] + diam(G) \leq \gamma_{ddls}(G)$.

Theorem 2.3: For any connected (p,q) graph G, $\alpha_0 + \beta_0 + 1 \leq \gamma_{ddls}(G)$.

Proof: $C = \{ v_1, v_2, \dots, v_n \} \subseteq V(G)$ be the minimal set of vertices with $dist(u,v) \geq 2$ for all $u, v \in C$, covers all the edges in G. Clearly, $|C| = \alpha_0(G)$. Further let $K = \{ v_1, v_2, \dots, v_n \} \subseteq V(G)$ be the maximum set of vertices such that $dist(u,v) \geq 2$ and $N(u) \cap N(v) = x, \forall u, v \in K$ and $x \in V(G) - K$. Clearly, $|K| = \beta_0(G)$. Now by the definition of line subdivision graph, let $F = \{u_1, u_2, \dots, u_i\} \subseteq V[L(S(G))]$ be the set of vertices corresponding to the edges which are incident with all the vertices of $S(G)$. Let $D^d = \{u_1, u_2, \dots, u_m\} \subseteq F$ be the set of vertices which is minimal double dominating set and covers all the vertices in line subdivision graph. Clearly D^d itself is a γ_{ddls}-set of G. Therefore it follows that $|C| \cup |K| \cup 1 \leq |D^d|$ and hence $\alpha_0 + \beta_0 + 1 \leq \gamma_{ddls}(G)$.

Theorem 2.4: If D^d is line subdivision double dominating set of a graph G, then $\frac{2p}{\delta(G)+1} \leq |D^d|$.

Proof: Let $D^d = \{u_1, u_2, \dots, u_i\}$ be line subdivision double dominating set of G and let t denote the number of edges joining the vertices of D^d to the vertices of $V[L(S(G))] - D^d$. Then $t = 2|V[L(S(G))] - D^d|$. By definition of double dominating set, every vertex u of D^d has exactly one neighbour in D^d. Thus $t = \sum_{u \in D^d} \deg(u) - 1$. So $|D^d|(\delta - 1) \geq t = 2|V(G) - D^d|$. Hence $\frac{2p}{\delta(G)+1} \leq |D^d|$.

Theorem 2.5: For any connected (p,q) graph$G$, $\gamma[L(S(G))] + i[L(S(G))] \leq \gamma_{ddls}(G)$. Equality hold if $G \cong P_3$.

Proof: Let $D = \{ v_1, v_2, \dots, v_n \} \subseteq V[L(S(G))]$ be the set of vertices which covers all the vertices in $L[S(G)]$. Then D is a minimal γ-set of $L[S(G)]$. Further if the subgraph $< D >$ contains the set of vertices v_i, $1 \leq i \leq n$ such that $\deg(v_i) = 0$. Then D itself is an independent dominating set of $L[S(G)]$. Otherwise $S = D' \cup I$ where $D' \subseteq D$ and $I \subseteq V[L(S(G))] - D$ forms a minimal independent dominating set of $L[S(G)]$. Since $V[L(S(G))] = E[S(G)]$ and let $I = \{u_1, u_2, \dots, u_n\} \subseteq V[L(S(G))]$ be the end vertices in $L[S(G)]$ such that any vertex $u \in V[L(S(G))] - D^d$ has at lest two neighbours in D^d and $|N[u] \cup D^d| \geq 2$. Clearly D^d forms a minimal $\gamma_{ddls}(G)$-set of G. So that $|D| \cup |S| \leq |D^d|$ and hence it gives $\gamma[L(S(G))] + i[L(S(G))] \leq \gamma_{ddls}(G)$.

Theorem 2.6: For any connected (p,q) graph$G$ with $p \geq 2$, $\left\lceil \frac{1+\Delta(G)}{p-q} \right\rceil \leq \gamma_{ddls}(G)$.

Proof: For any connected graph $p - q \leq 1$ and $\gamma_{ddls}(G) \geq 2$. Also for any graph G, $1 + \Delta(G) \geq 2$. It follows that $\left\lceil \frac{1+\Delta(G)}{p-q} \right\rceil \leq \gamma_{ddls}(G)$.

Theorem 2.7: For any connected (p,q) graph G with $p \geq 2, 2p - 2q \leq \gamma_{ddls}(G)$.

Proof: $D^d = \{ v_1, v_2, \dots, v_n \}$ be the minimal set of vertices which covers all the vertices in $L[S(G))]$. Suppose for any

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vertex $v \in V[L(S(G))] - D^d$ is adjacent to at least two vertices of D^d, clearly D^d forms a double dominating set of $L[S(G))]$. Let any vertex $v \in D^d$ which is not adjacent to any vertex of $V[L(S(G))] - D^d$. Then $2q \geq |D^d| + 2|V(G) - D^d|$ it gives $2q \geq |D^d| + 2p - 2|D^d|$. This implies $|D^d| \geq 2p - 2q$. Hence $2p - 2q \leq \gamma_{ddls}(G)$.

Theorem 2.8: For any connected (p,q) graph G, $\frac{2p+q-1}{2} \leq \gamma_{ddls}(G)$.

Proof: Let $D^d \subseteq V[L(S(G))]$ be a γ_{ddls}-set of G. Since $< V[L(s(G))] - D^d >$ is disconnected,

$$q \leq |D^d - V| + |D^d - V| + 1$$

$$\leq 2|D^d| - 2p + 1 \text{ it implies that}$$

$$2p + q - 1 \leq 2|D^d|. \text{ Hence } \frac{2p+q-1}{2} \leq \gamma_{ddls}(G).$$

Theorem 2.9: For any connected (p,q) graph G with $p \geq 2$ vertices, $\delta(G) + 1 \leq \gamma_{ddls}(G)$.

Proof: Let $D^d = \{ v_1, v_2, \dots, v_n \} \subseteq L[S(G))]$ be a γ_{ddls}-set of G. Then there exists a vertex $u \in D^d$ such that u is not a adjacent to any vertex of $V[L(s(G))] - D^d$. Thus $\deg(u) \leq (\gamma_{ddls}G) - 1$. Since $\delta(G) \leq \deg(u)$ implies that $\delta(G) + 1 \leq \gamma_{ddls}(G)$.

III. UPPER BOUNDS FOR $\gamma_{ddls}(G)$.

We establish upper bounds for $\gamma_{ddls}(G)$ in terms of elements of G.

Theorem 3.1: For any connected (p,q) graph G, $\gamma_{ddls}(G) \leq p + \left\lceil \frac{diam(G)}{2} \right\rceil$.

Proof: Let $J = \{e_1, e_2, \dots, e_n \} \subseteq E(G)$ be the edge set constituting the longest path between two distinct vertices $u, v \in V(G)$ such that $d(u,v) = diam(G)$. Since $V[L(S(G))] = E[S(G)]$ there exists a vertex set $D^d = \{ v_1, v_2, \dots, v_n \}$ such that any vertex $v \in V[L(S(G))] - D^d$ is adjacent to at least two vertices of D^d and $|N[v] \cap D^d| \geq 2$ it follows $|D^d| \geq 2$. We know that the diametric path includes at least two vertices. This implies that $2|D^d| \leq 2p + diam(G)$. Clearly implies that $\gamma_{ddns}(G) \leq p + \left\lceil \frac{diam(G)}{2} \right\rceil$.

Theorem 3.2: For any connected (p,q) graph G, $p + \Delta(G) \leq \gamma_{ddls}(G) + \gamma(G)$.

Proof: Let $C = \{ v_1, v_2, \dots, v_k \} \subseteq V(G)$ be the set of vertices with $\deg(v) \geq 2, \forall\, v \in C$. Then there exists at least one vertex $v \in C$ such that $\deg(v) = \Delta(G)$. Now without loss of generality in $L[S(G)]$, since $V[L(S(G))] = E[S(G)]$, there exists a set $D^d = \{u_1, u_2, \dots, u_m\} \subseteq V[L(S(G))]$ in $L[S(G)]$ covers all the vertices of $L[S(G)]$ such that any vertex $u \in V[L(S(G))] - D^d$ is adjacent to at least two vertices of D^d. Clearly D^d is a minimal double dominating set of $L[S(G)]$. It follows that $|D^d| \cup |D| \geq p + \Delta(G)$ which implies that $\gamma_{ddls}(G) + \gamma(G) \geq p + \Delta(G)$.

Theorem 3.3: For any connected (p,q) graph G, $\gamma_{ddls}(G) \leq diam(G) + \gamma(G) + \alpha_0$.

Proof: Let $C = \{ v_1, v_2, \dots, v_n \} \subseteq V(G)$ be the minimum set of vertices which covers all the edges in G with $|C| = \alpha_0(G)$. Further there exists an edge set $J' \subseteq J$, where J is the set of edges which are incident with the vertices of C, constituting the longest path in G such that $|J'| = diam(G)$. Let $D = \{ v_1, v_2, \dots, v_n \} \subseteq C$ be the minimal set of vertices which covers all the vertices in G, clearly D forms a minimal dominating set of G. Now in $L[S(G)]$, let $F = \{u_1, u_2, \dots, u_n\} \subseteq V[L(S(G))]$ and let $D^d = \{u_1, u_2, \dots, u_m\} \subseteq F$ such that any vertex $u \in V[L(S(G))] - D^d$ is adjacent to at least two vertices of D^d and $|N[u] \cap D^d| \geq 2$. Clearly D^d forms a minimal $\gamma_{ddls}(G)$-set of G. Therefore it follows that $|D^d| \leq |J'| \cup |D| \cup |C|$ and hence $\gamma_{ddls}(G) \leq diam(G) + \gamma(G) + \alpha_0$.

Theorem 3.4: For any nontrivial tree T, $\gamma_{ddls}(G) \leq p + m$, m is the number of cutvertices in T.

Proof: Let $A = \{ v_1, v_2, \dots, v_n \}$ be the set of all cutvertices in T with $|A| = m$. Suppose $C \subseteq V[S(G)]$, $\deg(v_i) \geq 2, \forall\, v_i \in C, 1 \leq i \leq n$ be the set of vertices in $S(G)$ and let $J = \{ e_1, e_2, \dots, e_k \}$ be the set of edges which are incident to the vertices of C. Now in $L(S(T))$ let $I = \{u_1, u_2, \dots, u_n\} \subseteq V[L(S(T))]$ be the set of vertices with $\deg(u_i) \geq 2, 1 \leq i \leq n$. Then $D^d = I \cup F'$ where $F = \{ u_1, u_2, \dots, u_m \} \subseteq J$ in $L(S(T))$ corresponding to the edges of J form a double dominating set of $L(S(T))$. Clearly it follows that $|I \cup F'| \leq p \cup |A|$ and hence $\gamma_{ddls}(G) \leq p + m$.

Theorem 3.5: For any connected (p,q) graph G with $p \geq 3, p + 1 \leq \gamma_{ddls}(G) \leq p + 2$.

Proof: Let $D^d = \{ v_1, v_2, \dots, v_n \}$ be a minimal line subdivision dominating set of G. Then every vertex in $V[L(S(G))] - D^d$ is dominated by at least two vertices in D^d. Therefore $2 \leq p + 1 \leq |D^d|$. This implies that $|V[L(S(G))] - D^d| \geq 0$ it gives $|V[L(S(G))]| \geq |D^d|$. Since $|V(G)| \leq |V[L(S(G))]|$. Thus $p + 1 \leq \gamma_{ddls}(G) \leq p + 2$.

Theorem 3.6: For any connected (p,q) graph G, $\gamma_{ddls}(G) \leq p + \left\lceil \frac{\Delta(G)}{2} \right\rceil$.

Proof: Let v be a vertex of degree $\Delta(G)$. Let F be the set of independent edges in $< N(v) >$. Let $D^d \subseteq V[L(S(G))]$ be a γ_{ddls}-set of G. Since $|F| \leq \left\lfloor \frac{\Delta(G)}{2} \right\rfloor$, therefore $|D^d| \leq |V(G) \cup N(v) - F|$

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$\leq p + \Delta(G) - \left\lfloor \frac{\Delta(G)}{2} \right\rfloor$

$\leq p + \left\lceil \frac{\Delta(G)}{2} \right\rceil$. Hence the result.

Theorem 3.7: For any connected (p,q) graph G, $\gamma_{ddls}(G) \leq p + q - \delta(G)$.

Proof: Let $D^d \subseteq V[L(S(G))]$ be a line subdivision double dominating set of G such that any vertex $u \in V[L(S(G))] - D^d$ has at least two neighbours in D^d. Therefore D^d be a γ_{ddls}-set of G. Suppose there exists a vertex $u \in D^d$ adjacent to vertices of D^d. Thus $\leq |D^d - V(G)| + \deg(u) \geq |D^d - V(G)| + \delta(G)$. This implies that $\gamma_{ddls}(G) \leq p + q - \delta(G)$.

Theorem 3.8: For any connected (p,q) graph G with $p \geq 2$, $2 \leq \gamma_{ddls}(G) \leq 2q$.

Proof: Let D^d be a minimum line subdivision double dominating set of G. Then

$|D^d| \leq |V(G) \cup E(G)| - 1$

$\leq p + q - (p - q) \leq 2q$. Hence the result.

IV. NORDHAUS-GADDUM TYPE RESULTS

Theorem 4.1: For any connected (p,q) graph G with $p \geq 3$ vertices,

(I) $\gamma_{ddls}(G) + \gamma_{ddls}(\bar{G}) \leq 2p + 4$.

(II) $\gamma_{ddls}(G) . \gamma_{ddls}(\bar{G}) \leq p^2 + 4p + 4$.

V. CONCLUSION

Domination in graph is one of the major research area in graph theory. Currently many interesting and important research area taking place in this area. Double domination is a particular type of domination and the double domination in graphs is relative new research area and hence there is a wide scope for studies in this particular area of domination theory. In this paper, we establish some upper and lower bounds on $\gamma_{ddls}(G)$. Further, its relation with other different dominating parameters are investigated. Nordhaus-Gaddum type results are also obtained for this parameter.

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