Line Subdivision Double Domination in Graphs

M. H. Muddebihal, Suhas P. Gade

1Department of Mathematics Gulbarga University, Kulburgi, Karnataka, India.
2Department of Mathematics, Sangameshwar College, Solapur, Maharashtra, India.
mhmuddebihal@gmail.com, suhaspanduranggade@gmail.com

ABSTRACT - Let $S(G)$ be the subdivision graph of $G$. The line graph of $S(G)$, $L(S(G))$ is a graph whose vertices correspond to the edges of $S(G)$ and two vertices in $L[S(G)]$ are adjacent if and only if corresponding edges in $S(G)$ are adjacent. A subset $D^d$ of $V[L(S(G))]$ is double domination set of $L[S(G)]$ if for every vertex $v \in V[L(S(G))], |N[v] \cap D^d| \geq 2$, that is $v$ is in $D^d$ and has at least one neighbour in $D^d$ or $v$ is in $V[L(S(G))] - D^d$ and has at least two neighbours in $D^d$. The line subdivision double domination number $\gamma_{dds}(G)$ is a minimum cardinality of the line subdivision double dominating set of $G$ and is denoted by $\gamma_{dds}(G)$. In this paper, we establish some upper and lower bounds on $\gamma_{dds}(G)$ in terms of the vertices, edges and other different parameters of $G$ and not in terms of the elements of $L[S(G)]$. Further, its relation with other different dominating parameters is also obtained. The main deal of this paper is to apply a probabilistic approach to obtain new bounds for line subdivision double domination parameter and to study their relationship with other different domination parameters of different graph valued functions.

SUBJECT CLASSIFICATION NUMBER: AMS – 05C69, 05C70.

KEYWORD: Line Subdivision Graph/Dominating set/Double domination.

I. INTRODUCTION

In this paper, all the graphs considered here are simple, finite, non-trivial, undirected and connected. The vertex set and edge set of graph $G$ are denoted by $V(G) = p$ and $E(G) = q$ respectively. The terms not defined here are used in the sense of Harary [9]. The neighbourhood of a vertex $v \in V$ is defined by $N(v) = \{ u \in V/uv \in E \}$. The close neighbourhood of a vertex $v$ is $N[v] = N(u) \cup \{ v \}$. The order $|V(G)|$ of $G$ is denoted by $p$. A vertex cover in a graph $G$ is a set of vertices that covers all the edges of $G$. The vertex covering number $\alpha_G$ is the minimum cardinality of a vertex cover in $G$. A set of vertices in a graph $G$ is called independent set if no two vertices are adjacent. The vertex independence number $\beta_G$ is the maximum cardinality of an independent set of vertices. A set $D$ of vertices in a graph $G$ is called dominating set of $G$ if every vertex in $V - D$ is adjacent to some vertex in $D$. The domination number of $G$, denoted by $\gamma(G)$ is the minimum cardinality of a dominating set. The domination in graphs with many variations is now well studied in graph theory. A thorough study of domination appears in [14]. Let $S(G)$ be the subdivision graph of $G$. The line graph of $S(G)$ is a graph whose vertices correspond to the edges of $S(G)$ and two vertices in $L[S(G)]$ are adjacent if and only if corresponding edges in $S(G)$ are adjacent. A subset $D^d$ of $V[L(S(G))]$ is double dominating set of $L[S(G)]$ if for every vertex $v \in V[L(S(G))]$, $|N[v] \cap D^d| \geq 2$, that is $v$ is in $D^d$ and has at least one neighbour in $D^d$ or $v$ is in $V[L(S(G))] - D^d$ and has at least two neighbours in $D^d$.

Observation 1: Let $G$ be a graph that admits line subdivision double domination set $D^d$. Then $|D^d| \geq p + \delta(G)$ if and only if $G = P_n$ with $p \geq 3$.

Observation 2: For any connected $(p, q)$ graph $G$, $p - \gamma_{dds}(G) \leq 0$.

Observation 3: For any connected $(p, q)$ graph $G$, $\gamma_{dds} \leq \gamma_{dds}(G)$. 

II. LOWER BOUNDS FOR $\gamma_{dds}(G)$.

We establish lower bound for $\gamma_{dds}(G)$ in terms of elements of $G$.

**Theorem 2.1:** For any connected $(p, q)$ graph $G$ with $p \geq 4$, $\gamma_{dds}(G) \geq \gamma_{ct}(L(G)) + 2$.

**Proof:** Let $A = \{v_1, v_2, \ldots, v_n\} \subseteq V(G)$ be the minimal set of vertices which covers all the vertices in $G$. Now in $L(G)$, $C = \{v_1, v_2, \ldots, v_n\}$ be the minimal set of vertices such that any vertex of $C$ has at least two neighbours in $L(G)$. Clearly, $|C| = 2$. Therefore it gives $\gamma_{ct}(L(G)) + 2 = \gamma_{ct}(L(G)) + \gamma_{ct}(G)$.

**Theorem 2.2:** For any connected $(p, q)$ graph $G$, $\gamma_{ct}(L(G)) + \text{diam}(G) \leq \gamma_{dds}(G)$.

**Proof:** Let $E^* = \{e_1, e_2, \ldots, e_k\} \subseteq E(G)$ be the minimal set of edges which constitute the longest path between any two distinct vertices $u, v \in V(G)$, such that $d(u, v) = \text{diam}(G)$. Now let $A = \{v_1, v_2, \ldots, v_n\} \subseteq V(G)$ be the set of all non end vertices in $G$. Further, let $F = \{\{v_1, v_2, \ldots, v_n\}\}$ be the set of edges which are incident to the vertices of $A$. Now in $L(G)$, suppose $I = \{u_1, u_2, \ldots, u_n\} \subseteq V(L(G))$ be the set of vertices with $\text{deg}(u_i) = 1, 1 \leq i \leq n$. Then $I^* = I \cup C^*$ where $C^* = \{u_1, u_2, \ldots, u_n\} \subseteq C$ in $L(G)$ corresponding to the edges of $C$ from a double dominating set of $L(G)$. Therefore it follows that $|D^*| \leq |D| \cup |I|$. Hence $\gamma_{dds}(G) \leq \gamma_{ct}(L(G)) + \gamma_{ct}(G)$.

**Theorem 2.3:** For any connected $(p, q)$ graph $G$, $\alpha_0 + \beta_0 + 1 \leq \gamma_{dds}(G)$.

**Proof:** $C = \{v_1, v_2, \ldots, v_n\} \subseteq V(G)$ be the minimal set of vertices with $\text{dist}(u, v) \geq 2$ for all $u, v \in C$, covers all the edges in $G$. Clearly, $|C| = \alpha_0(G)$. Further let $K = \{v_1, v_2, \ldots, v_n\} \subseteq V(G)$ be the maximum set of vertices such that $\text{dist}(u, v) \geq 2$ and $N(u) \cap N(v) = x, \forall u, v \in K$ and $x \in V(G) - K$. Clearly, $|K| = \beta_0(G)$. Now by the definition of line subdivision graph, let $F = \{u_1, u_2, \ldots, u_m\} \subseteq V(L(S(G)))$ be the set of vertices corresponding to the edges which are incident with all the vertices of $S(G)$. Let $D^* = \{u_1, u_2, \ldots, u_m\} \subseteq F$ be the set of vertices which is minimal double dominating set and covers all the vertices in line subdivision graph. Clearly $D^*$ itself is a $\gamma_{dds}$-set of $G$. Therefore it follows that $|C| \cup |K| \cup 1 \leq |D^*|$ and hence $\alpha_0 + \beta_0 + 1 \leq \gamma_{dds}(G)$.

**Theorem 2.4:** If $D^*$ is line subdivision double dominating set of a graph $G$, then $2p - |D^*| \leq |D^*|$.

**Proof:** Let $D^* = \{u_1, u_2, \ldots, u_m\} \subseteq V(L(S(G)))$ be line subdivision double dominating set of $G$ and let $t$ denote the number of edges joining the vertices of $D^*$ to the vertices of $V(L(S(G))) - D^*$. Then $t = 2|V(L(S(G))) - D^*|$. By definition of double dominating set, every vertex $u$ of $D^*$ has exactly one neighbour in $D^*$. Thus $t = \sum_{u \in D^*} \text{deg}(u) - 1$. So $|D^*| \geq 2$. Hence $2p \geq |D^*|$.

**Theorem 2.5:** For any connected $(p, q)$ graph $G$, $\gamma_{L(S(G)))} + i[L(S(G))) \leq \gamma_{dds}(G)$. Equality hold if $G \cong P_2$.

**Proof:** Let $D = \{v_1, v_2, \ldots, v_n\} \subseteq V(L(S(G)))$ be the set of vertices which covers all the vertices in $L[S(G)]$. Then $D$ is a minimal $\gamma$-set of $L[S(G)]$. Further if the subgraph $G < D$ contains the set of vertices $v_1, 1 \leq i \leq n$ such that $\text{deg}(v_i) = 0$. Then $D$ itself is an independent dominating set of $L[S(G)]$. Otherwise $S = D \cup I$ where $D \subseteq D$ and $I \subseteq V[L(S(G))] - D$ forms an independent dominating set of $L[S(G)]$. So that $|D \cup I| \leq |D^*|$ and hence it gives $\gamma_{L(S(G))} + i[L(S(G))] \leq \gamma_{dds}(G)$.

**Theorem 2.6:** For any connected $(p, q)$ graph $G$ with $p \geq 2$, $\gamma_{dds}(G) \leq \gamma_{dds}(G)$.

**Proof:** For any connected graph $p - q \leq 1$ and $\gamma_{dds}(G) \leq 2$. Also for any graph $G$, $1 + \Delta(G) \geq 2$. It follows that $\gamma_{dds}(G) \leq \gamma_{dds}(G)$.

**Theorem 2.7:** For any connected $(p, q)$ graph $G$ with $p \geq 2, 2p - 2q \leq \gamma_{dds}(G)$.

**Proof:** $D^* = \{v_1, v_2, \ldots, v_n\} \subseteq V(G)$ be the minimal set of vertices which covers all the vertices in $L[S(G)]$. Suppose for any
vertex \( v \in V[L(S(G))] - D^d \) is adjacent to at least two vertices of \( D^d \), clearly \( D^d \) forms a double dominating set of \( L[S(G)] \). Let any vertex \( v \in D^d \) which is not adjacent to any vertex of \( V[L(S(G))] - D^d \). Then \( 2q \geq |D^d| + 2|V(G) - D^d| \) it gives \( 2q \geq |D^d| + 2p - 2|D^d| \). This implies \( |D^d| \geq 2p - 2q \). Hence \( 2p - 2q \leq \gamma_{dats}(G) \).

**Theorem 2.8:** For any connected \((p, q)\) graph \( G \), \( \frac{2p+q-1}{2} \leq \gamma_{dats}(G) \).

**Proof:** Let \( D^d \subseteq V[L(S(G))] \) be a \( \gamma_{dats} \)-set of \( G \). Since \( V[L(S(G))] - D^d \) is disconnected,

\[
q \leq |D^d - V| + |D^d - V| + 1 \\
\leq 2|D^d - D| - 2p + 1 \text{ it implies that} \\
2p + q - 1 \leq 2|D^d|. \text{ Hence} \frac{2p+q-1}{2} \leq \gamma_{dats}(G).
\]

**Theorem 2.9:** For any connected \((p, q)\) graph \( G \) with \( p \geq 2 \) vertices, \( \delta(G) + 1 \leq \gamma_{dats}(G) \).

**Proof:** Let \( D^d = \{ v_1, v_2, ..., v_n \} \subseteq L[S(G)] \) be a \( \gamma_{dats} \)-set of \( G \). Then there exists a vertex \( u \in D^d \) such that \( u \) is not adjacent to any vertex \( V[L(S(G))] - D^d \). Thus \( \deg(u) \leq \gamma_{dats}(G) - 1 \). Since \( \delta(G) \leq \deg(u) \) implies that \( \delta(G) + 1 \leq \gamma_{dats}(G) \).

**III. UPPER BOUNDS FOR \( \gamma_{dats}(G) \).**

We establish upper bounds for \( \gamma_{dats}(G) \) in terms of elements of \( G \).

**Theorem 3.1:** For any connected \((p, q)\) graph \( G \),

\[
\gamma_{dats}(G) \leq p + \left\lceil \frac{diam(G)}{2} \right\rceil.
\]

**Proof:** Let \( J = \{ e_1, e_2, ..., e_k \} \subseteq E(G) \) be the edge set constituting the longest path between two distinct vertices \( u, v \in V(G) \) such that \( d(u, v) = diam(G) \). Since \( V[L(S(G))] = E(S(G)) \) there exists a vertex set \( D^d = \{ v_1, v_2, ..., v_n \} \) such that any vertex \( v \in V[L(S(G))] \) and let \( D^d \) is adjacent to at least two vertices of \( D^d \) and \( |N[v] \cap D^d| \geq 2 \) it follows \( |D^d| \geq 2 \). We know that the diametric path includes at least two vertices. This implies that \( 2|D^d| \leq 2p + diam(G) \). Clearly implies that \( \gamma_{dats}(G) \leq p + \left\lceil \frac{diam(G)}{2} \right\rceil \).

**Theorem 3.2:** For any connected \((p, q)\) graph \( G \), \( p + \Delta(G) \leq \gamma_{dats}(G) + \gamma(G) \).

**Proof:** Let \( C = \{ v_1, v_2, ..., v_k \} \subseteq V(G) \) be the set of vertices with \( \deg(v) \geq 2 \), \( \forall v \in C \). Then there exists at least one vertex \( v \in C \) such that \( \deg(v) = \Delta(G) \). Now without loss of generality in \( L[S(G)] \), since \( V[L(S(G))] = E(S(G)) \), there exists a set \( D^d = \{ u_1, u_2, ..., u_m \} \subseteq V[L(S(G))] \) in \( L[S(G)] \) covers all the vertices of \( L[S(G)] \) such that any vertex \( u \in V[L(S(G))] \) - \( D^d \) is adjacent to at least two vertices of \( D^d \). Clearly \( D^d \) is a minimal double dominating set of \( L[S(G)] \). It follows that \( |D^d| + |D| \geq p + \Delta(G) \) which implies that \( \gamma_{dats}(G) + \gamma(G) \geq p + \Delta(G) \).

**Theorem 3.3:** For any connected \((p, q)\) graph \( G \),

\[
\gamma_{dats}(G) \leq diam(G) + \gamma(G) + a_0.
\]

**Proof:** Let \( C = \{ v_1, v_2, ..., v_n \} \subseteq V(G) \) be the minimum set of vertices which covers all the edges in \( G \) with \( |C| = a_0 \). Further there exists an edge set \( J \subseteq J \), where \( J \) is the set of edges which are incident with the vertices of \( C \), constituting the longest path in \( G \) such that \( |J| = \text{diam}(G) \). Let \( D = \{ v_1, v_2, ..., v_n \} \subseteq C \) be the minimal set of vertices which covers all the vertices in \( G \). Clearly \( D \) forms a minimal dominating set of \( G \). Now in \( L[S(G)] \), let \( F = \{ u_1, u_2, ..., u_m \} \subseteq V[L(S(G))] \) and let \( D^d = \{ u_1, u_2, ..., u_m \} \subseteq F \) such that any vertex \( u \in V[L(S(G))] - D^d \) is adjacent to at least two vertices of \( D^d \) and \( |N[u] \cap D^d| \geq 2 \). Clearly \( D^d \) forms a minimal \( \gamma_{dats}(G) \)-set of \( G \). Therefore it follows that \( |D^d| \leq |J| \cup |D| \cup |C| \) and hence \( \gamma_{dats}(G) \leq \text{diam}(G) + \gamma(G) + a_0 \).

**Theorem 3.4:** For any nontrivial tree \( T \), \( \gamma_{dats}(G) \leq p + m \), \( m \) is the number of cutvertices in \( T \).

**Proof:** Let \( A = \{ v_1, v_2, ..., v_n \} \) be the set of all cutvertices in \( T \) with \( |A| = m \). Suppose \( C \subseteq V[S(G)] \), \( \deg(v) \geq 2 \), \( \forall v \in C, 1 \leq i \leq n \) be the set of vertices in \( S(G) \) and let \( J = \{ e_1, e_2, ..., e_k \} \) be the set of edges which are incident to the vertices of \( C \). Now in \( L(S(T)) \), let \( I = \{ u_1, u_2, ..., u_n \} \subseteq V[L(S(G))] \) be the set of vertices with \( \deg(u) \geq 2, 1 \leq i \leq n \). Then \( D^d = I \cup F \) where \( F = \{ u_1, u_2, ..., u_m \} \subseteq J \) in \( L(S(T)) \) corresponding to the edges of \( J \) form a double dominating set of \( L(S(T)) \). Clearly it follows that \( |I \cup F| \leq p + |A| \) and hence \( \gamma_{dats}(G) \leq p + m \).

**Theorem 3.5:** For any connected \((p, q)\) graph \( G \) with \( p \geq 3, p + 1 \leq \gamma_{dats}(G) \leq p + 2 \).

**Proof:** Let \( D^d = \{ v_1, v_2, ..., v_n \} \) be a minimal line subdivision dominating set of \( G \). Then every vertex in \( V[L(S(G))] - D^d \) is dominated by at least two vertices in \( D^d \). Therefore \( 2 \leq p + 1 \leq |D^d| \). This implies that \( |V[L(S(G))] - D^d| \geq 0 \) it gives \( |V[L(S(G))]| \geq |D^d| \). Since \( |V(G)| \leq |V[L(S(G))]| \). Thus \( p + 1 \leq \gamma_{dats}(G) \leq p + 2 \).

**Theorem 3.6:** For any connected \((p, q)\) graph \( G \),

\[
\gamma_{dats}(G) \leq p + \left\lceil \frac{\Delta(G)}{2} \right\rceil.
\]

**Proof:** Let \( v \) be a vertex of degree \( \Delta(G) \). Let \( F \) be the set of independent edges in \( < N(v) > \). Let \( D^d \subseteq V[L(S(G))] \) be a \( \gamma_{dats} \)-set of \( G \). Since \( |F| \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil \), therefore \( |D^d| \leq |V(G)| \cup N(v) - F |
\[
    \leq p + \Delta(G) - \left\lfloor \frac{\Delta(G)}{2} \right\rfloor \leq p + \frac{\Delta(G)}{2} \text{ Hence the result.}
\]

**Theorem 3.7:** For any connected \((p, q)\) graph \(G\), 
\(\gamma_{\text{data}}(G) \leq p + q - \delta(G)\).

**Proof:** Let \(D^d \subseteq V[L(S(G))]\) be a line subdivision double dominating set of \(G\) such that any vertex \(u \in V[L(S(G))]\) \(\setminus D^d\) has at least two neighbours in \(D^d\). Therefore \(D^d\) be a \(\gamma_{\text{data}}\)-set of \(G\). Suppose there exists a vertex \(u \in D^d\) adjacent to vertices of \(D^d\). Thus \(\leq |D^d - V(G)| + \deg(u) \geq |D^d - V(G)| + \delta(G)\). This implies that \(\gamma_{\text{data}}(G) \leq p + q - \delta(G)\).

**Theorem 3.8:** For any connected \((p, q)\) graph \(G\) with 
\(p \geq 2, 2 \leq \gamma_{\text{data}}(G) \leq 2q\).

**Proof:** Let \(D^d\) be a minimum line subdivision double dominating set of \(G\). Then
\[
|D^d| \leq |V(G) \cup E(G)| - 1 \leq p + q - (p - q) \leq 2q. \text{ Hence the result.}
\]

**IV. NORDHAUS-GADDUM TYPE RESULTS**

**Theorem 4.1:** For any connected \((p, q)\) graph \(G\) with 
\(p \geq 3\) vertices,

1. \(\gamma_{\text{data}}(G) + \gamma_{\text{data}}(\overline{G}) \leq 2p + 4\).
2. \(\gamma_{\text{data}}(G) \cdot \gamma_{\text{data}}(\overline{G}) \leq p^2 + 4p + 4\).

**V. CONCLUSION**

Domination in graph is one of the major research area in graph theory. Currently many interesting and important research area taking place in this area. Double domination is a particular type of domination and the double domination in graphs is relative new research area and hence there is a wide scope for studies in this particular area of domination theory. In this paper, we establish some upper and lower bounds on \(\gamma_{\text{data}}(G)\). Further, its relation with other different dominating parameters are investigated. Nordhaus-Gaddum type results are also obtained for this parameter.

**REFERENCES**


