On Sums and Product of k-J-EP Matrices

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Abstract **Necessary and sufficient conditions are determined for a sum of k-J-EP matrices to be k-J-EP. The sum and parallel summable of k-J-EP matrices to be k-J-EP. Also conditions are given under which a matrix of rank r is a product of k-J-EP_r matrices to be k-J-EP_r.**

***.Keywords** — k -Hermitian, J -Hermitian and permutation matrix.*

I. Introduction

Through out, we shall deal with $C_{n\times n}$ the space of $n\times n$ complex matrices. Let C_n be the space of complex n – tuples. For $A\in C_{n\times n}$, let A^T, A^* denote the transpose, conjugate transpose of A. Let A^- be a generalized inverse of A and $A^\dagger$ be the Moore-penrose inverse of A [10]. A matrix A is called EP_r if $\rho(A)=r$ and $R(A)=R(A^*)$ (or) $N(A)=N(A^*)$ where $\rho(A)$ denotes the rank of A; $N(A)$ and $R(A)$ denote the null space and range space of A respectively [7].let k - be a fixed product of disjoint transpositions in $S_n=\{1,2,\cdots n\}$. Since k is involutory, it can be verified that the associated permutation matrix K satisfy the following:

$$K=K^T=K^{-1} \text{ and } k(x)=Kx,\ K^2=I.$$

$$J=J^T=J^{-1} \text{ and } J(x)=Jx.$$

where J is an invertible Hermitian matrix. We make an additional assumption that $J^2=I$. A matrix $A=(a_{ij})\in C_{n\times n}$ is k -Hermitian if $a_{ij}=a_{k(j),k(i)}$ for $i,j=1,2\cdots n$. A theory for k -Hermitian matrices is developed in [4]. For, $x=(x_1,x_2\cdots x_n)^T\in C_n$. let us define the function $k(x)=(x_{k(1)},x_{k(2)}\cdots x_{k(n)})^T\in C_n$. A matrix $A\in C_{n\times n}$, is said to be if it satisfies the condition equivalently $N(A)=N(KA^{[*]}K)=N(KJA^*JK)$ [3]. Moreover, the indefinite matrix product of two matrices A and B of sizes $m\times n$ and $n\times l$ complex matrices respectively, is denoted to be the matrix by $A^{[*]}$ $A\circ B=AJ_nB$. The adjoint of A, denoted by $A^{[*]}$. Where $A^{[*]}=JA^*J$. A is said to be k-J-EP_r, if A is k-J-EP and of rank r. For further properties of k-J-EP matrix one may refer [8].

In section 1, we give necessary and sufficient conditions for sums of k-J-EP matrices to be k-J-EP. In section 2, it is shown that sum , parallel summable k-J-EP matrices are k-J-EP. In section 3, we explore the conditions that the product of k-J-EP_r matrices are k-J-EP_r.

Theorem 1.1:

Let A_i (i=1 to m) be k-J-EP matrices. Then $A=\sum_{i=1}^{m}A_i$ is k-J-EP iff any one of the following equivalent conditions hold:

(i) $N(A)\subseteq N(A_i)$ for each i

(ii) $rk\begin{bmatrix}A_1\\A_2\\\vdots\\A_m\end{bmatrix}=rk(A)$

Proof :

(i) $\Leftrightarrow$ (ii)

$N(A)\subseteq N(A_i)$ for each i implies

$N(A)\subseteq\bigcap N(A_i)$. Since

$N(A)=N(\sum A_i)\subseteq N(A_1)\cap N(A_2)......\cap N(A_m)$,

it follows that $N(A)\subseteq\bigcap N(A_i)$.

Hence, $N(A)=\cap N(A_i)=N\begin{bmatrix}A_1\\A_2\\\vdots\\A_m\end{bmatrix}$.

 DOI : 10.18231/2454-9150.2018.0723

Therefore, $rk(A) = rk\begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix}$ and (ii) holds.

Conversely, since $N\begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix} = \cap N(A_i) \subseteq N(A)$,

$rk\begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix} = rk(A)$. $N(A) = \bigcap N(A_i)$. Hence,

$N(A) \subseteq N(A_i)$ for each i and (i) holds. Since each A_i is k - J - EP, $N(A_i) = N(A_i^* JK)$ for each i, Now $N(A) \subseteq N(A_i)$ for each i, This implies that

$N(A) \subseteq N(A_i) = \cap N(A_i^* JK) \subseteq N(A^* JK)$.

Therefore $rk(A) = rk(A_i^* JK)$.

Hence $N(A) = N(A^* JK)$. Thus A is k - J - EP. Hence the theorem.

Remark 1.2:

In particular, if A is non singular the conditions automatically hold and is k - J - EP. Theorem (1.1) fails if we relax the conditions on the A_i.

Example 1.3:

Let k=(1 2) the associated permutation matrix Therefore

(i) Let $K = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$KJA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is EP.

Therefore, A is k - J - EP.

(ii) Let $K = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

$B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

$KJB = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ is not EP.

Therefore, B is not k - J - EP.

(iii) Let $K = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

$A+B = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ is not EP and

$KJ(A+B) = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ is not EP.

Therefore, $(A+B)$ is not k - J - EP. However, $N(A+B) \subseteq N(A^* JK) \subseteq N(A)$ and $N(A+B) \subseteq N(B^* JK) \subseteq N(B)$. Moreover $rk\begin{bmatrix} A \\ B \end{bmatrix} = rk(A+B)$.

Remark 1.4:

If rank is additive , that is $rk(A) = \sum rk(A_i)$, $R(A_i) \bigcap R(A_j) = \{0\}$, i $\neq$ j which implies that $N(A) \subseteq N(A_i)$ for each i $\Rightarrow N(A) \subseteq N(A_i^* JK)$ for each i. Hence A is k - J - EP. The conditions given in theorem (1.1) are weaker than the condition of rank additivity can be seen the following example.

Example 1.5:

Let

$K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, J = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$

, $A+B = \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix}$.

 DOI : 10.18231/2454-9150.2018.0723

Hence A, B and $(A+B)$ are k - J - EP. Conditions (i) and (ii) in theorem (1.1) hold.
But $rk(A+B)\neq rk(A)+rk(B)$.

Theorem 1.6:

Let A_i (i=1 to m) be k - J - EP matrices such that $A_i^*A_j=0$ then $A=\sum A_i$ is k - J - EP.

Proof :

Since $\sum_{i\neq j} A_i^*A_j=0$

$$A^*A=(\sum A_i)^*(\sum A_i)=(\sum A^*_i)(\sum A_i)$$
$$=\sum A^*_iA_i.$$
$$N(A)=N(A^*A)=N(\sum A^*_iA_i)$$
$$=N\left(\begin{bmatrix}A_1\\A_2\\\vdots\\A_m\end{bmatrix}^*\begin{bmatrix}A_1\\A_2\\\vdots\\A_m\end{bmatrix}\right)=N\left(\begin{bmatrix}A_1\\A_2\\\vdots\\A_m\end{bmatrix}\right)$$
$$=N(A_1)\bigcap N(A_2)\bigcap..........\bigcap(A_m).$$
$$=N(A_1^*JK)\cap N(A_2^*JK)\cap..........\cap N(A_m^*JK).$$

Hence $N(A)\subseteq N(A_i^*JK)$ for each i.

$N(A)=N(A_i)$ for each i. Now, A is k - J - EP follows from Theorem (1.1).

Remark 1.7:

Theorem (1.6) fails if we relax the condition that $A_i^{'}s$ are left. For let

$$K=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}, J=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}, A=\begin{pmatrix}1&1&0\\1&1&0\\0&0&0\end{pmatrix}$$
$$, B=\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}$$

$(i)KJA=\begin{pmatrix}-1&-1&0\\1&1&0\\0&0&0\end{pmatrix}$ is not EP. Therefore, A is not k - J - EP.

$(ii)KJB=\begin{pmatrix}0&0&-1\\0&1&0\\0&0&0\end{pmatrix}$ is not EP. Therefore, B is not k - J - EP.

Let $A=\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}, B=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$. Therefore,

$(A+B)=\begin{pmatrix}0&2&0\\1&0&1\\1&0&0\end{pmatrix}$ is not EP.

$(iii)KJ(A+B)=\begin{pmatrix}0&2&0\\1&0&1\\1&0&0\end{pmatrix}$ is not EP. Therefore, $(A+B)$ is not k - J - EP.

Remark 1.8:

The condition given in theorem (1.6) implies those in theorem (1.1) but not conversely. This can be seen by the following example.

Example 1.9:

$$K=\begin{pmatrix}0&1\\1&0\end{pmatrix}, J=\begin{pmatrix}1&0\\0&1\end{pmatrix}, A=\begin{pmatrix}0&0\\1&0\end{pmatrix},$$

$B=\begin{pmatrix}0&-1\\-1&0\end{pmatrix}$ A and B are k - J - EP matrices.

$$N(A+B)=N(KJ(A+B))=N(KJA+KJB)$$
$$\subseteq N(KJA)=N(A)$$

Therefore $N(A+B)\subseteq N(A)$. Also

$N(A+B)\subseteq N(\text{B})$. But $A^*B+B^*A\neq 0$.

Remark 1.10:

The conditions given in theorem (1.1) and theorem (1.6) are only sufficient for the sum of k - J - EP matrices to be k - J - EP, but not necessary is illustrated by the following example.

Example 1.11:

Let $A=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $B=\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$,

$K=\begin{pmatrix}0&1\\1&0\end{pmatrix}$ and $J=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$

A and B are k - J - EP_2. Neither the conditions in theorem (1.1) nor in theorem (1.6) hold. However, $(A+B)$ is k - J - EP. If A and B are k - J - EP matrices, $A^*=H_1KJAJK$ and $B^*=H_2KJBJK$. Where H_1 and H_2 are non singular $n\times n$ matrices. If $H_1=H_2$ then $A^*+B^*=H_1KJ(A+B)JK$ $\Rightarrow(A+B)^*=H_1KJ(A+B)JK\Rightarrow(A+B)$ is k - J - EP. If (H_1-H_2) is non singular, then the above conditions are also necessary for the sum of k - J - EP

 DOI : 10.18231/2454-9150.2018.0723

matrices to be k-J-EP is given in the in the following theorem.

Theorem 1.12:

Let $A^* = H_1KJAJK$ and $B^* = H_2KJBJK$ such that $(H_1 - H_2)$ is non singular and K be a permutation matrix, then $(A+B)$ is k-J-EP $\Leftrightarrow N(A+B) \subseteq N(B)$, where 'k' be the fixed transposition whose associative permutation matrix is K.

Proof :

Since $A^* = H_1KJAJK$ and $B^* = H_2KJBJK$, by a known theorem, A and B are k-J-EP matrices. Since $N(A+B) \subseteq N(B)$, by theorem (1.1), $(A+B)$ is k-J-EP. Conversely, let us assume that $(A+B)$ is k-J-EP. There exists a non singular matrix G such that $(A+B)^* = GKJ(A+B)JK$

$\Rightarrow A^* + B^* = GKJ(A+B)JK$

$\Rightarrow H_1KJAJK + H_2KJBJK = GKJ(A+B)JK$

$\Rightarrow (H_1KJA + H_2KJB)JK = GKJ(A+B)JK$

$\Rightarrow H_1KJA + H_2KJB = GKJA + GKJB$

$\Rightarrow H_1KJA - GKJA = GKJB - H_2KJB$

$\Rightarrow (H_1 - G)KJA = (G - H_2)KJB$

$\Rightarrow LKJA = MKJB$. Where $L = H_1 - G$ and $M = G - H_2$

Now, $(L+M)(KJA) = LKJA + MKJA$

$= MKJB + MKJA = MKJ(B+A) = MKJ(A+B)$

and $(L+M)KJB = LKJ(A+B)$. By hypothesis, $L + M = H_1 - G + G - H_2 = H_1 - H_2$ is non singular. Therefore,

$N(A+B) \subseteq N(MKJ(A+B)) = N((L+M)KJA)$

$= N(LKJA + MKJA) = N(KJA) = N(A)$. Therefore, $N(A+B) \subseteq N(A)$. Also,

$N(A+B) \subseteq N(LKJ(A+B)) = N((L+M)KJB)$

$= N(KJB) = N(B)$. Therefore, $N(A+B) \subseteq N(B)$. Thus $(A+B)$ is k-J-$EP \Rightarrow N(A+B) \subseteq N(A)$ and $N(A+B) \subseteq N(B)$. Hence the theorem.

Remark 1.13:

The condition $(H_1 - H_2)$ to be a non singular is essential in theorem (1.12) is illustrated in the following example.

Example 1.14:

$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$ are both k-J-EP matrices for $K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $J = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Further $A^* = A = KJAJK$ and $B^* = B = KJBJK$ $\Rightarrow H_1 - H_2 = I$. $(A+B) = \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix}$ is also k-J-EP. But $N(A+B) \not\subset N(A)$ (or) $N(A+B) \not\subset N(B)$. Thus theorem (1.12) fails.

We note that when $k(i) = i$, $J(i) = i$, for $i = 1$ to n the permutation matrix K and J reduces to I and theorem (1.1), theorem (1.6) and theorem (1.12) reduce to result to found in [4].

II. Parallel Summable k-J-EP Matrices

Here, it is shown that, sum and parallel sum of parallel summable k-J-EP matrices are k-J-EP. First, we shall quote the definition and some properties of parallel summable matrices [10] which are used in this section.

Definition 2.1:

A and B are said to be parallel summable (p.s) if $N(A+B) \subseteq N(B)$ and $N(A+B)^* \subseteq N(B^*)$ (or) equivalently $N(A+B) \subseteq N(A)$ and $N(A+B)^* \subseteq N(A^*)$.

Definition 2.2:

If A and B are parallel summable then parallel sum of A and B denoted by $A \mp B$ is defined as $A \mp B = A(A+B)^- B$. The product $A(A+B)^- B$ is invariant for all choices of generalized inverse $(A+B)^-$ of $(A+B)$ under the conditions that A and B are parallel summable [6].

Properties 2.3:

Let A and B be a pair of parallel summable (p.s) matrices. Then the following hold:

p.1 $A \mp B = B \mp A$

p.2 A^* and B^* are p.s and $(A \mp B)^* = A^* \mp B^*$

p.3 If U is non singular then UA and UB are p.s and $UA \mp UB = U(A \mp B)$

p.4 $R(A \mp B) = R(A) \cap R(B)$;
$R(A \mp B) = N(A) \mp N(B)$

p.5 $(A \mp B) \mp C = A \mp (B \mp C)$

 DOI : 10.18231/2454-9150.2018.0723

if all the parallel sum operations involved are defined.

Lemma 2.4:

Let A and B be k - J - EP matrices. Then A and B are p.s $\Leftrightarrow N(A+B) \subseteq N(A)$

Proof:

A and B are parallel summable $\Rightarrow N(A+B) \subseteq N(A)$ follows from definition (2.1).

Conversely, if $N(A+B) \subseteq N(A)$, then $N(KJA+KJB) \subseteq N(KJA)$. Also $N(KJA+KJB) \subseteq N(KJB)$. Since KJA and KJB are EP matrices and $N(KJA+KJB) \subseteq N(KJA)$ and $N(KJA+KJB) \subseteq N(KJB)$, by theorem (1.1), $(KJA+KJB)$ is EP .Hence

$N(KJA+KJB)^* = N(KJA+KJB)$
$= N(KJA) \cap N(KJB) = N(KJA)^* \cap N(KJB)^*$.

Therefore, $N(KJA+KJB)^* \subseteq N(KJA)^*$ and $N(KJA+KJB)^* \subseteq N(KJB)^*$.Also, $N(KJA+KJB) \subseteq N(KJA)$ by hypothesis. Hence, by Definition (1.1), KJA and KJB are p.s.

$N(KJA+KJB) \subseteq N(KJA)$
$= N(KJ(A+B)) \subseteq N(KJA)$
$\Rightarrow N(A+B)^* \subseteq N(B^*)$.

Also $N(KJA+KJB)^* \subseteq N(KJA)^*$
$\Rightarrow N(KJ(A+B))^* \subseteq N(KJA)^*$
$\Rightarrow N(A+B)^* \subseteq N(A^*)$. Therefore, A and B are p.s.

Hence the theorem.

Remark 2.5:

Lemma (2.4) fails if we relax the condition that A and B are k - J - EP. Let $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$. Let the associated permutation matrix be $K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $J = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. A is k - J - EP. B is not k - J - EP.

$N(A+B) \subseteq N(A)$ and $N(A+B) \subseteq N(B)$, but $N(A+B)^* \not\subset N(A^*)$; $N(A+B)^* \not\subset N(B^*)$. Hence, A and B are not parallel summable.

Theorem 2.6:

Let A and B be p.s. k - J - EP matrices. Then $(A \mp B)$ and $(A+B)$ are k - J - EP.

Proof :

Since A and B are p.s k - J - EP matrices , by Lemma (2.4), $N(A+B) \subseteq N(A)$ and $N(A+B) \subseteq N(B)$. $N(KJ(A+B)) \subseteq N(KJA)$ and $N(KJ(A+B)) \subseteq N(KJB)$.

$N(KJA+KJB) \subseteq N(KJA)$ and $N(KJA+KJB) \subseteq N(KJB)$. Therefore, $KJ(A+B) = (KJA+KJB)$ is EP. Then, $(A+B)$ is k - J - EP follows from theorem (1.1). Since A and B are p.s k - J - EP matrices. Therefore, $R(KJA)^* = R(KJA)$ and $R(KJB)^* = R(KJB)$.

$R(KJA \mp KJB)^* = R((KJA)^* \mp (KJB)^*)$
$= R((KJA)^*) \cap R((KJB)^*) = R(KJA) \cap R(KJB)$
$= R(KJA \mp KJB)$.

Thus $(KJA \mp KJB)$ is EP. Implies that $KJ(A \mp B)$ is EP. This implies that $A \mp B$ is k - J - EP. Thus $A \mp B$ is k - J - EP .whenever A and B are k - J - EP.

Hence the theorem.

Remark 2.7:

The sum and parallel sum of p.s k - J - EP matrices are k - J - EP.

Corollary 2.8:

Let A and B be k - J - EP matrices such that $N(A+B) \subseteq N(B)$. If C is k - J - EP commuting with both A and B, then $C(A+B)$ and $C(A \mp B) = (CA \mp CB)$ are k - J - EP.

Proof :

A and B are k - J - EP with $N(A+B) \subseteq N(B)$. By Theorem (1.1), $(A+B)$ is k - J - EP .Now KJA, KJB and $KJ(A+B)$ are EP. Since C commutes with A, B and $(A+B)$, KJC Commutes with KJA, KJB and $KJ(A+B)$ and by known theorem, $KJ(CA)$, $KJ(CB)$ and $KJ(C(A+B))$ are EP. Therefore, CA, CB and $C(A+B)$ are k - J - EP. Now by theorem (2.6), $(CA \mp CB)$ is k - J - EP. By p.3 (properties (2.3)), $KJ(C(A \mp B)$ is EP implies that $C(A \mp B)$ is k - J - EP.

 DOI : 10.18231/2454-9150.2018.0723

Corollary 2.9:

Let A and B be p.s EP matrices. Then $A \mp B$ and $(A+B)$ are J - EP .

Corollary 2.10:

Let A and B be p.s J - EP matrices. Then $A \mp B$ and $(A+B)$ are EP . When $J = I$.

Corollary 2.11:

Let A and B be p.s k - J - EP matrices. Then $A \mp B$ and $(A+B)$ are k - EP . when $K = I$.

III. Product of k - J - EP matrices

It is well known that the product of nonsingular matrices is non singular. In general, the product of symmetric, Hermitian, normal and EP matrices need not be respectively symmetric, Hermitian, normal and EP matrices [1,2]. Similarly, the product of k - J - EP matrices need not be k - J - EP . For instance, Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $J = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. A is k - J - EP_1, B is k - J - EP_1 . $AB = \begin{pmatrix} -1 & 0 \\ -1 & 0 \end{pmatrix}$ is not k - J - EP_1. In this section, we explore the conditions for the product of k - J - EP_r matrices to be k - J - EP_r [5,6,9]. Also, we study the question of when BA is k - J - EP_r , for k - J - EP_r matrices A , B and AB .

Theorem 3.1:

Let A_1 and A_n $(n>1)$ be k - J - EP_r matrices and let $A = A_1A_2A_3\cdots A_n$. Then the following statements are equivalent:

(i) A is k - J - EP_r .

(ii) $R(A_1) = R(A_n)$ and $rk(A) = r$.

(iii) $R(A_1^*) = R(A_n^*)$ and $rk(A) = r$.

Proof :

(i) $\Leftrightarrow$ (ii)

Since A_1 and A_n are k - J - EP_r , $R(A_1) = R(KJA_1^*)$, $R(A_n) = R(KJA_n^*)$. Since $A = A_1A_2A_3\cdots A_n$, $R(A) \subseteq R(A_1)$ and $rk(A) = rk(A_1) \Rightarrow R(A) = R(A_1)$. Also $A^* = A_n^* \cdots A_1^*$. $R(A^*) \subseteq R(A_n^*)$ and $rk(A) = rk(A_n) = r \Rightarrow rk(A^*) = rk(A_n^*) = r$.

Therefore, $R(A^*) = R(A_n^*)$

$\Rightarrow R(KJA^*) = R(KJA_n^*)$. Now,

A is k - J - EP_r $\Leftrightarrow R(A) = R(KJA^*)$ and

$rk(A) = r \Leftrightarrow R(A_1) = R(KJA_n^*)$ and $rk(A) = r$

$\Leftrightarrow R(A_1) = R(A_n)$ and $rk(A) = r$

(ii) $\Leftrightarrow$ (iii)

$R(A_1) = R(A_n) \Leftrightarrow R(KJA_1^*) = R(KJA_n^*)$

$\Leftrightarrow R(A_1^*) = R(A_n^*)$.

Hence the theorem.

Corollary 3.2:

Let A and B be k - J - EP_r matrices. Then AB is k - J - $EP_r \Leftrightarrow rk(AB) = r$ and $R(A) = R(B)$.

Proof :

AB is $EP_r \Leftrightarrow R(A) = R(B)$ and $rk(AB) = r$.

AB is k - $EP_r \Leftrightarrow R(KA) = R(KB)$ and

$rk(AB) = r \Leftrightarrow R(A) = R(B)$ and $rk(AB) = r$.

AB is k - J - $EP_r \Leftrightarrow R(KJA) = R(KJB)$ and

$rk(AB) = r \Leftrightarrow R(A) = R(B)$ and $rk(AB) = r$.

Example 3.3:

Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$

$J = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, KJA^*JK = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, KJB^*JK = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

. A is k - J - EP_1 and B is k - J - EP_1. $AB = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ is not k - J - EP_1. Here $rk(AB) = 1$ and $R(A) \neq R(B)$.

Example 3.4:

Let

$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$

$J = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, A is k - J - EP_1 and B is k - J - EP_1. $AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is not k - J - EP_1. Here $rk(AB) \neq 1$, $R(A) = R(B)$.

 DOI : 10.18231/2454-9150.2018.0723

Theorem 3.5:

Let $rk(AB) = rk(B) = r_1$ and $rk(BA) = rk(A) = r_2$. If AB, B are k - J - EP_{r_1} and A is k - J - EP_{r_2} then BA is k - J - EP_{r_2}.

Proof :

Since $rk(BA) = rk(A) = r_2$, it is enough to show that $N(BA) = N((BA)^* JK)$, to prove BA is k - J - EP_{r_2}.Now, $N(A) \subseteq N(BA)$ and $rk(BA) = rk(A) \Rightarrow N(A) = N(BA).$ Also, $N(B) \subseteq N(AB)$ and $rk(AB) = rk(B)$ $\Rightarrow N(B) = N(AB).$ Now

$N(BA) = N(A) = N(A^* JK) \subseteq N(B^* A^* JK) =$

$N(AB) = N(B) = N(B^* JK) \subseteq N((BA)^* JK)$.

Further, $rk(BA) = rk(BA)^* = rk((BA)^* JK)$

$\Rightarrow N(BA) = N((BA)^* JK).$ Thus, BA is k - J - EP_{r_2}.

Hence the theorem.

Lemma 3.6:

If A, B are k - J - EP_r matrices and AB has rank r , then BA has rank r .

Proof :

Since $rk(AB) = rk(B) = r,$

$N(A) \cap N(B^*)^{\perp} = \{0\}.$

By a known theorem $rk(AB) = rk(B)$

$-\dim(N(A) \bigcap N(B^*)^{\perp}).$ $N(A) \cap N(B^*)^{\perp} = \{0\}$

$\Rightarrow N(A) \cap N(BJK) = \{0\}$

$\Rightarrow N(A)^{\perp} \cap N(BJK) = \{0\}$

$\Rightarrow N(A^* JK)^{\perp} \cap N(BJK) = \{0\}$

Now, $rk(BA) = rk((BJK)(KJA)) =$

$rk(KJA) - \dim(N(BJK) \cap N(A^* JK)^{\perp}) =$

$rk(KJA) - 0 = rk(A) - 0 = r$.

Hence the lemma.

Theorem 3.7:

If A, B and AB are k - J - EP_r matrices then BA is k - J - EP_r.

Proof :

Since A, B and AB are k - J - EP_r matrices and $rk(AB) = r,$ by Lemma (3.5), $rk(BA) = r.$ Now the theorem follows from theorem (3.4), for $r_1 = r_2 = r.$

Corollary 3.8:

Let A, B be k - J - EP_r matrices. Then the following statements are equivalent:

(i) AB is k - J - EP_r

(ii) $(AB)^{\dagger}$ is k - J - EP_r

(iii) $A^{\dagger} B^{\dagger}$ is k - J - EP_r

(iv) $B^{\dagger} A^{\dagger}$ is k - J - EP_r

IV. CONCLUSION

These results may lead to study the Schur complement, partial ordering and generalization of inverses.

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 DOI : 10.18231/2454-9150.2018.0723