Generalized Hyers-Ulam Stability of a AQ Functional Equation

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Abstract **In this paper, the authors generalized 3-D additive-quartic functional equation of the form (1.3) is introduce in Fuzzy Normed Space. Further, stability of the functional equation verified by the generalized Hyers-Ulam method associated with direct and fixed point methods.**

Keywords — Additive functional equation, Quartic functional equation, Fixed Point, Hyers-Ulam Stability, Fuzzy Normed Space.

MSC: 39B52, 32B72, 32B82.

I. Introduction

In 1984, katsaras [5] defined a fuzzy norm on a linear space to construct a fuzzy structure on the space. Since then, some mathematicians have introduced several types of fuzzy norm in different points of view. In particular, we use the definition of a fuzzy normed space given[1] to exhibit a reasonable fuzzy version of stability for the cubic and quadratic functional equation in the fuzzy normed space.

The functional equation

$$f(x+y) = f(x) + f(y) \qquad (1.1)$$

is called the Cauchy additive functional equation and it is the most famous functional equation. Since $f(x) = kx$ is the solution of the functional equation (1.1).Then the functional equation

$$f(2x+y) + f(2x-y) = 4f(x+y) + 4f(x-y) + 24f(x) - 6f(y) \qquad (1.2)$$

is called the Quartic functional equation. It is easy to show that the function $f(x) = x^4$ satisfies the functional equation (1.2), which is called a quartic functional equation and every solution of the quartic functional equation is said to be a quartic function.

J. M. Rassias[8,9,10] was discussed about the linear mappings and T. M. Rassias[11] also investigate the linear mappings in banach space. Govindan et al.,[3,4,6,13] prescribed some stability of the various functional equations in various kind.

Some other papers of functional equation are used to develop this paper such as [2,7,12]. In this paper, the authors investigate the stability for the Additive-Quartic functional equation is of the form

$$\begin{aligned} & f(rx_1+r^2x_2+r^3x_3) + f(-rx_1+r^2x_2+r^3x_3) + f(rx_1-r^2x_2+r^3x_3) \\ & + f(rx_1+r^2x_2-r^3x_3) = 2\big[f(rx_1+r^2x_2) + f(r^2x_2+r^3x_3) \\ & + f(rx_1+r^3x_3) + f(rx_1-r^2x_2) + f(r^2x_2-r^3x_3) + f(rx_1-r^3x_3)\big] \\ & -2\big[r^4(f(x_1)+f(-x_1)) + r^8(f(x_2)+f(-x_2)) + r^{12}(f(x_3)+f(-x_3))\big] \\ & -\big[r(f(x_1)-f(-x_1)) + r^2(f(x_2)-f(-x_2)) + r^3(f(x_3)-f(-x_3))\big] \end{aligned} \qquad (1.3)$$

in fuzzy banach space using direct and fixed point method.

II .FUZZY STABILITY RESULTS

In this section, the authors present the basic definition in fuzzy normed space.

Definition 2.1 Let x be a real linear space. A function $F : X \times R \to [0,1]$ is said to be a fuzzy norm on X if for all $x, y \in X$ and all $p, q \in R$

(N1) $F(x,c) = 0$ for $c \le 0$;

(N2) $x = 0$ if and only if $F(x,c) = 1$ for all $c > 0$;

(N3) $F(cx,q) = F\left(x, \frac{q}{|c|}\right)$ if $c \ne 0$;

(N4) $F(x+y, p+q) \ge \min\{F(x,p), F(y,q)\}$;

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(N5) $F(x,.)$ is a non-decreasing function on R and $\lim_{q\to\infty} F(x,q)=1$;

(N6) for $x\neq 0$, $F(x,.)$ is continuous on R;

The pair (X,F) is called fuzzy normed linear space one may regard $F(x,q)$ as the truth value of the statement the norm of x is less than or equal to the real number q.

Definition 2.2 Let (X,F) be a fuzzy normed linear space. Let $\{x_n\}$ be a sequence in X. Then x_n is said to be convergent if there exists $x\in X$ such that $\lim_{n\to\infty}(x_n-x,q)=1$ for all $t>0$. In that case x is called the limit of the sequence x_n and we denote it by $F-\lim_{n\to\infty} x_n = x$.

Definition 2.3 A sequence $\{x_n\}$ be in x is called Cauchy if for each $\varepsilon>0$ and each $q>0$ there exists n_0 such that for all $n\geq n_0$ and all $\delta>0$, we have $F(x_{n+\delta}-x_n,q)>1-\varepsilon$

Definition 2.4 Every convergent sequence in fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and fuzzy normed space is called a fuzzy Banach space.

In section 3 and 4, assume that X, (Z,F) and (Y,F') are linear space, fuzzy normed space and fuzzy Banach space respectively. We define a function $R:X\to Y$ by

$$\begin{aligned}R(x_1,x_2,x_3) &= f(rx_1+r^2x_2+r^3x_3)+f(-rx_1+r^2x_2+r^3x_3)\\ &+f(rx_1-r^2x_2+r^3x_3)+f(rx_1+r^2x_2-r^3x_3)-2\Big[f(rx_1+r^2x_2)\\ &+f(r^2x_2+r^3x_3)+f(rx_1+r^3x_3)+f(rx_1-r^2x_2)\\ &+f(r^2x_2-r^3x_3)+f(rx_1-r^3x_3)\Big]+2\Big[r^4\big(f(x_1)+f(-x_1)\big)\\ &+r^8\big(f(x_2)+f(-x_2)\big)+r^{12}\big(f(x_3)+f(-x_3)\big)\Big]+\\ &\Big[r\big(f(x_1)-f(-x_1)\big)+r^2\big(f(x_2)-f(-x_2)\big)+r^3\big(f(x_3)-f(-x_3)\big)\Big]\end{aligned}$$

for all $x_1,x_2,x_3\in X$.

II. STABILITY OF THE FUNCTIONAL EQUATION (1.3) - DIRECT METHOD

In this section, we establish the stability of (1.3) in fuzzy Banach space using direct method.

Theorem 3.1 Let $\beta\in\{-1,1\}$. Let $\chi: X^3\to Z$ be a mapping with $0<\left(\frac{d}{r^4}\right)<1$

$$F'\left(\chi\left(r^{\beta k}x_1,r^{\beta k}x_2,r^{\beta k}x_3\right),\delta\right)\geq F'\left(d^{\beta}\chi(x,0,0),\delta\right) \tag{3.1}$$

for all $x\in X$ and all $\delta>0$, $d>0$ and

$$\lim_{k\to\infty} F'\left(\chi\left(r^{\beta k}x_1,r^{\beta k}x_2,r^{\beta k}x_3\right),r^{4\beta k}\delta\right)=1 \tag{3.2}$$

for all $x_1,x_2,x_3\in X$ and all $\delta>0$. Suppose that a function $R:X\to Y$ satisfies the inequality

$$F\left(R(x_1,x_2,x_3),\delta\right)\geq F'\left(\chi(x_1,x_2,x_3),\delta\right) \tag{3.3}$$

for all $r>0$ and $x_1,x_2,x_3\in X$ the limit

$$G(x)=F-\lim_{k\to\infty}\frac{f\left(r^{\beta k}x\right)}{r^{4\beta k}} \tag{3.4}$$

exists for all $x\in X$ and the mapping $G:X\to Y$ is a unique quartic mapping such that

$$F\left(f(x)-G(x),\delta\right)\geq F'\left(\chi(x,0,0),4\delta\left|r^4-d\right|\right) \tag{3.5}$$

for all $x\in X$ and for all $\delta>0$.

Proof. First assume that $\beta=1$. Replacing (x_1,x_2,x_3) by $(x,0,0)$, in (3.3), we have

$$F\left(\left(4f(rx)-4r^4f(x)\right),\delta\right)\geq F'\left(\chi(x,0,0),\delta\right) \tag{3.6}$$

for all $x\in X$ and for all $\delta>0$. Replacing x by r^kx in (3.6), we obtain

$$F\left(\frac{f\left(r^{k+1}x\right)}{r^4}-f\left(r^kx\right),\frac{\delta}{4r^4}\right)\geq F'\left(\chi\left(r^kx,0,0\right),\delta\right) \tag{3.7}$$

for all $x\in X$ and for all $\delta>0$. Using (3.1), (N3) in (3.7), we have

$$F\left(\frac{f\left(r^{k+1}x\right)}{r^4}-f\left(r^kx\right),\frac{\delta}{4r^4}\right)\geq F'\left(\chi\left(r^kx,0,0\right),\frac{\delta}{d^k}\right) \tag{3.8}$$

for all $x\in X$ and for all $\delta>0$, it is easy to verify from (3.8), that

$$F\left(\frac{f\left(r^{k+1}x\right)}{r^{4(k+1)}}-\frac{f\left(r^kx\right)}{r^{4k}},\frac{\delta}{4r^{4(k+1)}}\right)\geq F'\left(\chi\left(r^kx,0,0\right),\frac{\delta}{d^k}\right) \tag{3.9}$$

holds for all $x\in X$ and for all $\delta>0$. Replacing δ by $d^k\delta$ in (3.9)

$$F\left(\frac{f\left(r^{k+1}x\right)}{r^{4(k+1)}}-\frac{f\left(r^kx\right)}{r^{4k}},\frac{d^k\delta}{4r^{4(k+1)}}\right)\geq F'\left(\chi\left(r^kx,0,0\right),\delta\right) \tag{3.10}$$

for all $x\in X$ and for all $\delta>0$, it is easy to see that

$$\frac{f\left(r^{k+1}x\right)}{r^{4(k+1)}}-f(x)=\sum_{i=0}^{k-1}\left[\frac{f\left(r^{i+1}x\right)}{r^{4(i+1)}}-\frac{f\left(r^ix\right)}{r^{4i}}\right] \tag{3.11}$$

for all $x\in X$. From the equations (3.10) and (3.11), we get

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$$F\left(\frac{f\left(r^{k}x\right)}{r^{4k}}-f\left(x\right),\sum_{i=0}^{k-1}\frac{d^{i}\delta}{4r^{4(i+1)}}\right)$$
$$\geq \min\bigcup_{i=1}^{k-1}\left\{\frac{f\left(r^{i+1}x\right)}{r^{4(i+1)}}-\frac{f\left(r^{i}x\right)}{r^{4i}},\frac{d^{i}\delta}{4r^{4(i+1)}}\right\}$$
$$\geq \min\bigcup_{i=1}^{k-1}F'\left(\chi\left(x,0,0\right),\delta\right)$$
$$\geq F'\left(\chi\left(x,0,0\right),\delta\right) \qquad (3.12)$$

for all $x \in X$ and for all $\delta > 0$. Replacing x by $r^{m}x$ in (3.12) and using (3.1) and (N3), we obtain

$$F\left(\frac{f\left(r^{k+m}x\right)}{r^{4(k+m)}}-\frac{f\left(r^{m}x\right)}{r^{4m}},\sum_{i=0}^{m+k-1}\frac{d^{i}\delta}{4r^{4(i+1)}}\right)$$
$$\geq F'\left(\chi\left(x,0,0\right),\frac{\delta}{d^{m}}\right) \qquad (3.13)$$

for all $x \in X$ and for all $\delta > 0$. And all $m,k \geq 0$. Replacing δ by $d^{m}\delta$ in (3.13), we get

$$F\left(\frac{f\left(r^{k+m}x\right)}{r^{4(k+m)}}-\frac{f\left(r^{m}x\right)}{r^{4m}},\sum_{i=m}^{m+k-1}\frac{d^{i}\delta}{4r^{4(i+1)}}\right)$$
$$\geq F'\left(\chi\left(x,0,0\right),\delta\right) \qquad (3.14)$$

for all $x \in X$ and for all $\delta > 0$. And all $m,k \geq 0$. Using (N3) in (3.13), we have

$$F\left(\frac{f\left(r^{k+m}x\right)}{r^{4(k+m)}}-\frac{f\left(r^{m}x\right)}{r^{4m}},\delta\right)$$
$$\geq F'\left(\chi\left(x,0,0\right),\frac{\delta}{\sum_{i=m}^{m+k-1}\frac{d^{i}}{4r^{4(i+1)}}}\right) \qquad (3.15)$$

for all $x \in X$ and for all $\delta > 0$. And all $m,k \geq 0$. Since $0 < d < r^{4}$ and $\sum_{i=0}^{k}\left(\frac{d}{r^{4}}\right)^{i} < \infty$. The Cauchy criterion for convergence and (N5) implies that $\left\{\frac{f\left(r^{k}x\right)}{r^{4k}}\right\}$ is a Cauchy sequence in $\left(Y,F'\right)$ is a fuzzy Banach space. This sequence converges to some point $G\left(x\right)\in Y$ so one can define the mapping $G: X \to Y$ by $G\left(x\right)=F-\lim_{k\to\infty}\frac{f\left(r^{\beta k}x\right)}{r^{4\beta k}}$ for all $x \in X$. Letting $m=0$ in (3.15), we receive

$$F\left(\frac{f\left(r^{k}x\right)}{r^{4k}}-f\left(x\right),\delta\right)\geq F'\left(\chi\left(x,0,0\right),\frac{\delta}{\sum_{i=0}^{k-1}\frac{d^{i}}{4r^{4(i+1)}}}\right) \qquad (3.16)$$

for all $x \in X$. Letting $k \to \infty$ in (3.16) and using (N6), we have

$$F\left(f\left(x\right)-G\left(x\right),\delta\right)\geq F'\left(\chi\left(x,0,0\right),4\delta\left(r^{4}-d\right)\right)$$

for all $x \in X$ and for all $\delta > 0$. To prove G satisfies (1.3), replacing $\left(x_1,x_2,x_3\right)$ by $\left(r^{k}x_1,r^{k}x_2,r^{k}x_3\right)$ in (3.3), we get

$$F\left(\frac{1}{r^{4k}}R\left(r^{k}x_1,r^{k}x_2,r^{k}x_3\right),\delta\right) \qquad (3.17)$$
$$\geq F'\left(\chi\left(r^{k}x_1,r^{k}x_2,r^{k}x_3\right),r^{4k}\delta\right)$$

for all $\delta > 0$ and all $x_1,x_2,x_3 \in X$. Hence G satisfies the quartic functional equation (1.3). In order to prove $G\left(x\right)$ is unique. We let $G'\left(x\right)$ be another quartic functional equation satisfying (1.3) and (3.5). Hence

$$F\left(G\left(x\right)-G'\left(x\right),\delta\right)=F\left(\frac{G\left(r^{k}x\right)}{r^{4k}}-\frac{G'\left(r^{k}x\right)}{r^{4k}}\right)$$
$$\geq \min\left\{F\left(\frac{G\left(r^{k}x\right)}{r^{4k}}-\frac{f\left(r^{k}x\right)}{r^{4k}},\frac{\delta}{2}\right),F\left(\frac{f\left(r^{k}x\right)}{r^{4k}}-\frac{G'\left(r^{k}x\right)}{r^{4k}},\frac{\delta}{2}\right)\right\}$$
$$\geq F'\left(\chi\left(r^{k}x,0,0\right),\frac{4r^{4k}\delta\left(r^{4}-d\right)}{2}\right)$$
$$\geq F'\left(\chi\left(x,0,0\right),\frac{4r^{4k}\delta\left(r^{4}-d\right)}{2d^{k}}\right)$$

for all $x \in X$ and for $\delta > 0$. Since,

$$\lim_{k\to\infty}\frac{4r^{4k}\delta\left(r^{4}-d\right)}{2d^{k}}=\infty,$$

we obtain

$$F'\left(\chi\left(r^{k}x,0,0\right),\frac{4r^{4k}\delta\left(r^{4}-d\right)}{2d^{k}}\right)=1$$

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Thus $F\left(G(x)-G^{'}(x),\delta\right)=1$ for all $x\in X$ and for $\delta>0$. Hence $G(x)=G^{'}(x)$. Therefore $G(x)$ is unique. For $\beta=-1$, we can prove the result by a similar method. This completes the proof of the Theorem.

The following corollary is an immediate consequence of the Theorem 3.1, concerning the stability of (1.3).

Corollary 3.2 Suppose that the function $R: X\to Y$ satisfies the inequality

$$F\left(R(x_1,x_2,x_3),\delta\right)\ge\begin{cases}F^{'}(\varepsilon,\delta)\\ F^{'}\left(\varepsilon\sum_{i=1}^{3}\|x_i\|^{s},\delta\right),\\ F^{'}\left(\varepsilon\left(\sum_{i=1}^{3}\|x_i\|^{3s}+\prod_{i=1}^{3}\|x_i\|^{s}\right),\delta\right),\end{cases}$$

for all $x_1,x_2,x_3\in X$ and all $\delta>0$, where ε, s are constants. Then there exists a unique quartic mapping $G: X\to Y$ such that

$$F\left(f(x)-G(x),\delta\right)\ge\begin{cases}F^{'}\left(\varepsilon,4\delta\left|r^{4}-1\right|\right)\\ F^{'}\left(\varepsilon\|x\|^{s},4\delta\left|r^{4}-r^{s}\right|\right);\ s\ne 4\\ F^{'}\left(\varepsilon\|x\|^{3s},4\delta\left|r^{4}-r^{3s}\right|\right);\ s\ne\frac{4}{3}\end{cases}$$

for all $x\in X$ and for $\delta>0$.

Proposition 3.3 Let $\beta\in\{-1,1\}$. Let $\chi: X^{3}\to Z$ be a mapping with $0<\left(\frac{d}{r}\right)<1$

$$F^{'}\left(\chi\left(r^{\beta k}x_1,r^{\beta k}x_2,r^{\beta k}x_3\right),\delta\right)\ge F^{'}\left(d^{\beta}\chi(x,0,0),\delta\right)$$

for all $x\in X$ and all $\delta>0$, $d>0$ and

$$\lim_{k\to\infty}F^{'}\left(\chi\left(r^{\beta k}x_1,r^{\beta k}x_2,r^{\beta k}x_3\right),r^{\beta k}\delta\right)=1$$

for all $x_1,x_2,x_3\in X$ and all $\delta>0$. Suppose that a function $R: X\to Y$ satisfies the inequality

$$F\left(R(x_1,x_2,x_3),\delta\right)\ge F^{'}\left(\chi(x_1,x_2,x_3),\delta\right)$$

for all $r>0$ and $x_1,x_2,x_3\in X$ the limit

$$G(x)=F-\lim_{k\to\infty}\frac{f\left(r^{\beta k}x\right)}{r^{\beta k}}$$

exists for all $x\in X$ and the mapping $G: X\to Y$ is a unique additive mapping such that

$$F\left(f(x)-G(x),\delta\right)\ge F^{'}\left(\chi(x,0,0),2\delta|r-d|\right)$$

for all $x\in X$ and for all $\delta>0$.

Corollary 3.4 Suppose that the function $R: X\to Y$ satisfies the inequality

$$F\left(R(x_1,x_2,x_3),\delta\right)\ge\begin{cases}F^{'}(\varepsilon,\delta)\\ F^{'}\left(\varepsilon\sum_{i=1}^{3}\|x_i\|^{s},\delta\right),\\ F^{'}\left(\varepsilon\left(\sum_{i=1}^{3}\|x_i\|^{3s}+\prod_{i=1}^{3}\|x_i\|^{s}\right),\delta\right),\end{cases}$$

for all $x_1,x_2,x_3\in X$ and all $\delta>0$, where ε, s are constants. Then there exists a unique additive mapping $G: X\to Y$ such that

$$F\left(f(x)-G(x),\delta\right)\ge\begin{cases}F^{'}\left(\varepsilon,2\delta|r-1|\right)\\ F^{'}\left(\varepsilon\|x\|^{s},2\delta\left|r-r^{s}\right|\right);\ s\ne 1\\ F^{'}\left(\varepsilon\|x\|^{3s},2\delta\left|r-r^{3s}\right|\right);\ s\ne\frac{1}{3}\end{cases}\quad\text{f}$$

or all $x\in X$ and for $\delta>0$.

IV. Stability of the Functional Equation (1.3) – Fixed Point Method

In this section, the authors investigate the generalized Ulam-Hyers stability of the functional equation (1.3) in fuzzy normed space using fixed point method.

For to prove the stability result we define the following ψ_i is a constant such that

$$\psi_i=\begin{cases}r & if\ i=0\\ \frac{1}{r} & if\ i=1\end{cases}$$

and Ω is the set such that $\Omega=\{p\setminus p: X\to Y, p(0)=0\}$.

Theorem 4.1 Let $R: X\to Y$ be a mapping for which there exists a function $\chi: X^{3}\to Z$ with condition

$$\lim_{k\to\infty}F^{'}\left(\chi\left(\psi_i^{k}x_1,\psi_i^{k}x_2,\ldots,\psi_i^{k}x_n\right),\psi_i^{4k}\delta\right)=1\qquad(4.1)$$

for all $x_1,x_2,x_3\in X$ and $\delta>0$ and satisfying the inequality

$$F\left(R(x_1,x_2,x_3),\delta\right)\ge F^{'}\left(\chi(x_1,x_2,x_3),\delta\right)\qquad(4.2)$$

for all $x_1,x_2,x_3\in X$ and $\delta>0$. If there exists $L=L[i]$ such that the function $x\to\rho(x)$ has the property

$$F^{'}\left(L\frac{1}{\psi_i^{4}}\rho(\psi_i x),\delta\right)=F^{'}\left(\rho(x),\delta\right)\qquad(4.3)$$

for all $x\in X$ and $\delta>0$. Then there exists unique quartic function $G: X\to Y$ satisfying the functional equation (1.3) and

$$F\left(f(x)-G(x),\delta\right)\ge F^{'}\left(\frac{L^{1-i}}{1-L}\rho(x),\delta\right)$$

for all $x\in X$ and $\delta>0$.

Proof. Let d be a general metric on Ω, such that

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$$d(p,q)=\inf\{k\in(0,\infty)/F(p(x)-q(x),\delta)\geq F'(\rho(x),k\delta),\ x\in X,\delta>0\}.$$

It is easy to see that (Ω,d) is complete. Define $T:\Omega\to\Omega$ by $Tp(x)=\frac{1}{\psi_i^4}p(\psi_i x),\ \forall\, x\in X.$

For $p,q\in\Omega$, we get

$$d(p,q)=k\Rightarrow F(p(x)-q(x))\geq F'(\rho(x),k\delta)$$

$$\Rightarrow F\left(\frac{p(\psi_i x)}{\psi_i^4}-\frac{q(\psi_i x)}{\psi_i^4},\delta\right)\geq F'\left(\rho(\psi_i x),k\psi_i^4\delta\right) \quad (4.4)$$

$$\Rightarrow F(Tp(x)-Tq(x),\delta)\geq F'\left(\rho(\psi_i x),k\psi_i^4\delta\right)$$

$$\Rightarrow F(Tp(x)-Tq(x),\delta)\geq F'(\rho(x),kL\delta)$$

$$\Rightarrow d(Tp(x)-Tq(x),\delta)\geq kL$$

$$\Rightarrow d(Tp-Tq,\delta)\geq kd(p,q),\forall p,q\in\Omega.$$

Therefore T is strictly contractive mapping on Ω with Lipschitz constant L, replacing (x_1,x_2,x_3) by $(x,0,0)$ in (4.2), we get

$$F\left(4f(rx)-4r^4f(x),\delta\right)\geq F'(\chi(x,0,0),\delta) \quad (4.5)$$

for all $x\in X$ and $\delta>0$. Using (N3) in (4.5), we have

$$F\left(\frac{f(rx)}{r^4}-f(x),\delta\right)\geq F'\left(\frac{1}{4r^4}\chi(x,0,0),\delta\right) \quad (4.6)$$

for all $x\in X$ and $\delta>0$ with the help of (4.3), when $i=0$. It follows from (4.6) that

$$\Rightarrow F\left(\frac{f(rx)}{r^4}-f(x),\delta\right)\geq F'(L\rho(x),\delta)$$

$$\Rightarrow d(Tf(x),\delta)\geq L=L^1=L^{1-i} \quad (4.7)$$

Replacing x by $\frac{x}{r}$ in (4.5), we receive

$$F\left(f(x)-r^4f\left(\frac{x}{r}\right),\delta\right)\geq F'\left(\frac{1}{4}\chi\left(\frac{x}{r},0,0\right),\delta\right) \quad (4.8)$$

for all $x\in X$ and $\delta>0$ when $i=1$ it follows from (4.8), we arrive

$$\Rightarrow F\left(f(x)-r^4f\left(\frac{x}{r}\right),\delta\right)\geq F'(\rho(x),\delta)$$

$$\Rightarrow T(f-Tf)\leq 1=L^0=L^{1-i} \quad (4.9)$$

Then from (4.7) and (4.9), we get

$$\Rightarrow T(f,Tf)\leq L^{1-i}<\infty.$$

Now from the fixed point alternative in both cases it follows that there exists a fixed point G of T in Ω such that

$$G(x)=F-\lim_{k\to\infty}\frac{f(\psi^k x)}{\psi^{4k}} \quad (4.10)$$

for all $x\in X$ and $\delta>0$. Replacing (x_1,x_2,x_3) by $(\psi_i^k x_1,\psi_i^k x_2,\psi_i^k x_3)$ in (4.2), we get

$$F\left(\frac{1}{\psi_i^{4k}}R(\psi_i^k x_1,\psi_i^k x_2,\psi_i^k x_3),\delta\right)\geq F'\left(\chi(\psi_i^k x_1,\psi_i^k x_2,\psi_i^k x_3),\psi_i^{4k}\delta\right)$$

for all $\delta>0$ and all $x_1,x_2,x_3\in X$. By proceeding the some procedure in the theorem (3.7), we can prove the function $G:X\to Y$ is quartic and its satisfies the functional equation (1.3) by a fixed point alternative. Since G is unique fixed point of T in the set $\Delta=\{f\in\Omega/d(f,G)<\infty\}$. Therefore G is a unique function such that

$$F(f(x)-G(x),\delta)\geq F'(\rho(x),k\delta) \quad (4.11)$$

for all $x\in X$ and $\delta>0$. Again using the fixed point alternative, we get

$$d(f,G)\leq\frac{1}{1-L}d(f,Tf)$$

$$\Rightarrow d(f,G)\leq\frac{L^{1-i}}{1-L}$$

$$\Rightarrow F(f(x)-G(x),\delta)\geq F'\left(\rho(x)\frac{L^{1-i}}{1-L},\delta\right) \quad (4.12)$$

This completes the proof of the Theorem.

The following corollary is an immediate consequence of Theorem 4.1 concerning the stability of (1.3).

Corollary 4.2 Suppose that a function $R:X\to Y$ satisfies the inequality

$$F(R(x_1,x_2,x_3),\delta)\geq\begin{cases}F'(\varepsilon,\delta),\\ F'\left(\varepsilon\left\{\sum_{i=1}^{3}\|x_i\|^s\right\},\delta\right),\\ F'\left(\varepsilon\left\{\prod_{i=1}^{3}\|x_i\|^s+\sum_{i=1}^{3}\|x_i\|^{3s}\right\},\delta\right),\end{cases} \quad (4.13)$$

for all $x_1,x_2,x_3\in X$ and $\delta>0$, where ε,s are constants with $\varepsilon>0$. Then there exists an unique quartic function $G:X\to Y$ such that

$$F(f(x)-G(x),\delta)\geq\begin{cases}F'\left(\varepsilon,4\delta\left|r^4-1\right|\right)\\ F'\left(\varepsilon\|x\|^s,4\delta\left|r^4-r^s\right|\right);\ s\neq 4\\ F'\left(\varepsilon\|x\|^{3s},4\delta\left|r^4-r^{3s}\right|\right);\ s\neq\frac{4}{3}\end{cases} \quad (4.14)$$

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for all $x \in X$ and $\delta > 0$.

Proof. Setting

$$\chi(x_1,x_2,x_3) \le \begin{cases} \varepsilon, \\ \varepsilon\left\{\sum_{i=1}^{3}\|x_i\|^s\right\}, \\ \varepsilon\left\{\prod_{i=1}^{3}\|x_i\|^s + \sum_{i=1}^{3}\|x_i\|^{3s}\right\}, \end{cases}$$

for all $x_1,x_2,x_3 \in X$. Then

$$F'\left(\chi\left(\psi_i^k x_1,\psi_i^k x_2,\psi_i^k x_3\right),\psi_i^{4k}\delta\right)$$

$$= \begin{cases} F'\left(\varepsilon,\psi_i^{4k}\delta\right), \\ F'\left(\varepsilon\left\{\sum_{i=1}^{3}\|x_i\|^s\right\},\psi_i^{(4-s)k}\delta\right), \\ F'\left(\varepsilon\left\{\prod_{i=1}^{3}\|x_i\|^s + \sum_{i=1}^{3}\|x_i\|^{3s}\right\},\psi_i^{(4-3s)k}\delta\right) \end{cases}$$

$$= \begin{cases} \to 1 \text{ as } k \to \infty \\ \to 1 \text{ as } k \to \infty \\ \to 1 \text{ as } k \to \infty \end{cases}$$

i.e., (4.1) is holds. Since, we have

$$\rho(x) = \frac{1}{4}\chi\left(\frac{x}{r},0,0\right)$$

has the property

$$F'\left(L\frac{1}{\psi_i^4}\rho(\psi_i x),\delta\right) = F'(\rho(x),\delta)$$

for all $x \in X$ and $\delta > 0$. Hence

$$F'(\rho(x),\delta) = F'\left(\chi\left(\frac{x}{r},0,0\right),4\delta\right)$$

$$= \begin{cases} F'\left(\varepsilon,4r^4\delta\right) \\ F'\left(\varepsilon\|x\|^s,4r^s\delta\right) \\ F'\left(\varepsilon\|x\|^{3s},4r^{3s}\delta\right) \end{cases}$$

Now

$$F'\left(\frac{1}{\psi_i^4}\rho(\psi_i x),\delta\right) = \begin{cases} F'\left(\frac{\varepsilon}{\psi_i^4},4\delta\right) \\ F'\left(\frac{\varepsilon\|x\|^s\psi_i^s}{\psi_i^4 r^s},4\delta\right) \\ F'\left(\frac{\varepsilon\|x\|^{3s}\psi_i^{3s}}{\psi_i^4 r^{3s}},4\delta\right) \end{cases}$$

$$= \begin{cases} \psi_i^{-4}\rho(x) \\ \psi_i^{s-4}\rho(x) \\ \psi_i^{3s-4}\rho(x) \end{cases}$$

for all $x \in X$. Now from the following cases for the conditions

$L = r^{-4}$ if $i = 0$ and $L = r^4$ if $i = 1$

$L = r^{s-4}$ for $s > 4$ if $i = 0$ and $L = r^{4-s}$ for $s < 4$ if $i = 1$

$L = r^{3s-4}$ for $s > \frac{4}{3}$ if $i = 0$ and $L = r^{4-3s}$ for $s < \frac{4}{3}$ if $i = 1$

Case1. $L = r^{-4}$ *if* $i = 0$

$$F\left(f(x)-G(x),\delta\right) \le F'\left(\frac{L^{1-i}}{1-L}\rho(x),\delta\right) = F'\left(\frac{r^{-4}}{1-r^{-4}}\frac{\varepsilon}{4},\delta\right)$$

$$\le F'\left(\varepsilon,4\left(r^4-1\right)\delta\right)$$

Case2. $L = r^4$ *if* $i = 1$

$$F\left(f(x)-G(x),\delta\right) \le F'\left(\frac{L^{1-i}}{1-L}\rho(x),\delta\right) = F'\left(\frac{1}{1-r^4}\frac{\varepsilon}{4},\delta\right)$$

$$\le F'\left(\varepsilon,4\left(1-r^4\right)\delta\right)$$

Case 3. $L = r^{s-4}$ for $s > 4$ if $i = 0$

$$F\left(f(x)-G(x),\delta\right) \le F'\left(\frac{L^{1-i}}{1-L}\rho(x),\delta\right) = F'\left(\frac{r^{s-4}}{1-r^{s-4}}\frac{\varepsilon\|x\|^s}{4r^s},\delta\right)$$

$$\le F'\left(\varepsilon\|x\|^s,4\left(r^4-r^s\right)\delta\right)$$

Case 4. $L = r^{4-s}$ for $s < 4$ if $i = 1$

$$F\left(f(x)-G(x),\delta\right) \le F'\left(\frac{L^{1-i}}{1-L}\rho(x),\delta\right) = F'\left(\frac{1}{1-r^{4-s}}\frac{\varepsilon\|x\|^s}{4r^s},\delta\right)$$

$$\le F'\left(\varepsilon\|x\|^s,4\left(r^s-r^4\right)\delta\right)$$

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Case 5. $L = r^{3s-4}$ for $s > \frac{4}{3}$ if $i = 0$

$$F\left(f(x)-G(x),\delta\right) \le F'\left(\frac{L^{1-i}}{1-L}\rho(x),\delta\right) = F'\left(\frac{r^{3s-4}}{1-r^{3s-4}}\frac{\varepsilon\|x\|^{3s}}{4r^{3s}},\delta\right) \le F'\left(\varepsilon\|x\|^{3s},4\left(r^{4}-r^{3s}\right)\delta\right)$$

Case 6. $L = r^{4-3s}$ for $s < \frac{4}{3}$ if $i = 1$

$$F\left(f(x)-G(x),\delta\right) \le F'\left(\frac{L^{1-i}}{1-L}\rho(x),\delta\right) = F'\left(\frac{1}{1-r^{4-3s}}\frac{\varepsilon\|x\|^{3s}}{4r^{3s}},\delta\right) \le F'\left(\varepsilon\|x\|^{3s},4\left(r^{3s}-r^{4}\right)\delta\right)$$

Hence the proof is complete.

Proposition 4.3 Let $R: X \to Y$ be a mapping for which there exists a function $\chi: X^3 \to Z$ with condition

$$\lim_{k\to\infty} F'\left(\chi\left(\psi_i^k x_1,\psi_i^k x_2,\ldots,\psi_i^k x_n\right),\psi_i^k\delta\right) = 1$$

for all $x_1,x_2,x_3 \in X$ and $\delta > 0$ and satisfying the inequality

$$F\left(R\left(x_1,x_2,x_3\right),\delta\right) \ge F'\left(\chi\left(x_1,x_2,x_3\right),\delta\right)$$

for all $x_1,x_2,x_3 \in X$ and $\delta > 0$. If there exists $L = L[i]$ such that the function $x \to \rho(x)$ has the property

$$F'\left(L\frac{1}{\psi_i}\rho\left(\psi_i x\right),\delta\right) = F'\left(\rho(x),\delta\right)$$

for all $x \in X$ and $\delta > 0$. Then there exists unique additive function $G: X \to Y$ satisfying the functional equation (1.3) and

$$F\left(f(x)-G(x),\delta\right) \ge F'\left(\frac{L^{1-i}}{1-L}\rho(x),\delta\right)$$

for all $x \in X$ and $\delta > 0$.

Corollary 4.2 Suppose that a function $R: X \to Y$ satisfies the inequality

$$F\left(R\left(x_1,x_2,x_3\right),\delta\right) \ge \begin{cases} F'\left(\varepsilon,\delta\right), \\ F'\left(\varepsilon\left\{\sum_{i=1}^{3}\|x_i\|^{s}\right\},\delta\right), \\ F'\left(\varepsilon\left\{\prod_{i=1}^{3}\|x_i\|^{s}+\sum_{i=1}^{3}\|x_i\|^{3s}\right\},\delta\right), \end{cases}$$

for all $x_1,x_2,x_3 \in X$ and $\delta > 0$, where ε, s are constants with $\varepsilon > 0$. Then there exists an unique additive function $G: X \to Y$ such that

$$F\left(f(x)-G(x),\delta\right) \ge \begin{cases} F'\left(\varepsilon,2\delta\left|r-1\right|\right) \\ F'\left(\varepsilon\|x\|^{s},2\delta\left|r-r^{s}\right|\right);\ s \ne 1 \\ F'\left(\varepsilon\|x\|^{3s},2\delta\left|r-r^{3s}\right|\right);\ s \ne \frac{1}{3} \end{cases}$$

for all $x \in X$ and $\delta > 0$.

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