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A Parameter-Uniform Essentially First Order Convergent Numerical Method for a System of Singularly Perturbed Differential Equations of Reaction-Diffusion Type with Robin Boundary Conditions

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Abstract - **In this paper, a class of linear systems of singularly perturbed second order ordinary differential equations of reaction-diffusion type with Robin boundary conditions is considered. The components of the solution $\vec{u}$ of this system are smooth, whereas the components of $\vec{u}'$ exhibit boundary layers. A piecewise-uniform Shishkin mesh is introduced and is used in conjunction with a classical finite difference discretisation, to construct a numerical method for solving this problem. It is proved that the numerical approximations obtained with this method are essentially first order convergent uniformly with respect to all of the parameters. Numerical illustration is provided to support the theory.**

Keywords* — *Singular perturbations, boundary layers, system of differential equations, Robin boundary conditions, finite difference scheme, Shishkin meshes, parameter-uniform convergence.

I. INTRODUCTION

The following two point boundary value problem is considered for the system of singularly perturbed linear second order differential equations,

$$-E\vec{u}''(x) + A(x)\vec{u}(x) = \vec{f}(x), \quad x \in \Omega = (0,1) \qquad (1.1)$$

with

$$\vec{u}(0) - \vec{u}'(0) = \vec{\phi}_0, \quad \vec{u}(1) + \vec{u}'(1) = \vec{\phi}_1. \qquad (1.2)$$

Here $\vec{u}$ is a column n -vector, E and $A(x)$ are $n \times n$ matrices, $E = diag(\vec{\varepsilon}), \vec{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_n)$, with $0 < \varepsilon_i < 1$ for all $i = 1, \dots, n$. The parameters ε_i are assumed to be distinct and for convenience, the ordering $\varepsilon_1 < \cdots < \varepsilon_n$ is assumed.

The above problem can be rewritten in the operator form,

$$L\vec{u} = \vec{f} \text{ on } \Omega \ , \qquad (1.3)$$

$$\beta_0\vec{u}(0) = \vec{\phi}_0, \quad \beta_1\vec{u}(1) = \vec{\phi}_1, \qquad (1.4)$$

where the operators L, β_0, β_1 are defined by

$$L = -ED^2 + A, \quad \beta_0 = I - ID, \quad \beta_1 = I + ID$$

where I is the identity operator, $D = \frac{d}{dx}$ and $D^2 = \frac{d^2}{dx^2}$ are the first and second order differential operators.

For all $x \in [0,1]$, it is assumed that the components $a_{ij}(x)$ of $A(x)$ satisfy the inequalities

$$a_{ii}(x) > \sum_{j\neq i, j=1}^{n} |a_{ij}(x)| \text{ for } 1 \le i \le n \text{ and } a_{ij}(x) \le 0 \text{ for } i \neq j \qquad (1.5)$$

and for some α,

$$0 < \alpha < min_{x\in\bar{\Omega}, 1\le i\le n} \left(\sum_{j=1}^{n} a_{ij}(x)\right) \qquad (1.6)$$

It is also assumed, without loss of generality, that

$$\sqrt{\varepsilon_n} \le \frac{\sqrt{\alpha}}{4}. \qquad (1.7)$$

Further the functions a_{ij}, f_i are assumed to be in $C^{(2n-2)}(\bar{\Omega}), n > 1$, for $i, j = 1, \dots, n$ where $\bar{\Omega} = [0,1]$. From the above assumptions, (1.1),(1.2) has a solution $\vec{u} \in C(\bar{\Omega}) \cap C^{(2n)}(\Omega), n > 1$.

The reduced problem obtained by putting each $\varepsilon_i = 0, i = 1, \dots, n$, in the system (1.1) is the linear algebraic system,

$$A(x)\vec{u}_0(x) = \vec{f}(x)\,, \qquad (1.8)$$

where $\vec{u}_0(x) = \left(u_{0,1}(x), u_{0,2}(x), \dots, u_{0,n}(x)\right)^T$. The problem (1.1),(1.2) is singularly perturbed in the following sense. The solution $\vec{u}$ is expected to have the following layer pattern. Each component u_i for $i = 1, \dots, n$ is expected to exhibit weak twin layers at $x = 0$ and $x = 1$ of width $O(\sqrt{\varepsilon_n})$, while the components u_i for $i = 1, \dots, n-1$ have additional weak twin layers of width $O(\sqrt{\varepsilon_{n-1}})$, the components u_i for $i = 1, \dots, n-2$ have additional weak twin layers of width $O(\sqrt{\varepsilon_{n-2}})$ and so on.

The norms $\|y\|_D = sup_{x\in D}|y(x)|$ for any scalar–valued function y and domain D, and $\|\vec{y}\|_D = max_{1\le k\le n}\|y_k\|_D$ for any vector-valued function $\vec{y} = (y_1, y_2, \dots, y_n)^T$, are introduced. Throughout the paper, C denotes a generic positive constant, which is independent of x and of all singular perturbation and the discretization parameters. Furthermore, inequalities between vectors are understood in the componentwise sense.

For a general introduction to parameter-uniform numerical methods for singular perturbation problems, see [1],[2],[3] and [4]. In [5], a mixed Neumann-Robin boundary value problem for the Laplace operator in a smooth domain in R^2 is studied. The Robin condition

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contains a parameter $E$and tends to a Dirichlet condition as $\varepsilon \to 0$. A complete asymptotic expansion of the solution in powers of E is given. Sharp estimates in various Sobolev norms is given and in particular that there exist terms of order $O(E \log E)$ is shown. In [6], Nonlinear turning point problems is considered that admit boundary and/or interior layers at positions that are not determined a priori. For positive "viscosity" first order derivative terms are allowed in the boundary operator. Under certain conditions, shown in a sense to be sharp, the viscous limit of such problems is characterized and prove that they are identical to those limit solutions obtained from the pure Dirichlet problem. In [7], the conservative form of singularly perturbed ordinary differential equations with mixed boundary conditions is considered. A fitted mesh finite difference scheme is constructed for these problems. The scheme is shown to be uniformly convergent with respect to the perturbation parameter. A class of conservative difference schemes with uniform mesh are also considered. These difference schemes are proved to be first-order uniformly convergent.

II. ANALYTICAL RESULTS

The operator L satisfies the following maximum principle.

Lemma 2.1 Let $A(x)$ satisfy (1.5),(1.6). Let $\vec{\psi}$ be any vector-valued function in the domain of L such that $\beta_0\vec{\psi}\,(0) \geq \vec{0},\ \beta_1\vec{\psi}\,(1) \geq \vec{0}$. Then $L\vec{\psi}(x) \geq \vec{0}$ on $x \in \Omega$ implies that $\vec{\psi}(x) \geq \vec{0}$ on $x \in \bar{\Omega}$.

Proof : Let i^*, x^* be such that $\psi_{i^*}(x^*) = min_i\, min_{x\in\bar{\Omega}} \psi_i\,(x)$ and assume that the lemma is false.

Then, $\psi_{i^*}(x^*) < 0$. For $x^* = 0$, $(\beta_0\vec{\psi})_{i^*}(0) = \psi_{i^*}(0) - \psi'_{i^*}(0) < 0$ and for $x^* = 1$, $(\beta_1\vec{\psi})_{i^*}(1) = \psi_{i^*}(1) + \psi'_{i^*}(1) < 0$, contradicting the hypothesis. Therefore, $x^* \notin \{0,1\}$ and $\psi''_{i^*}(x^*) \geq 0$.

Thus,

$$(L\vec{\psi})_{i^*}(x^*) = -\varepsilon_{i^*}\psi''_{i^*}(x^*) + \sum_{j=1}^{n} a_{i^*j}(x^*)\,\psi_{i^*}(x^*) < 0,$$

which contradicts the assumption and proves the result for L.

Lemma 2.2 Let $A(x)$ satisfy (1.5),(1.6). Let $\vec{\psi}$ be any vector-valued function the domain of L, then for each $i,\ 1 \leq i \leq n$ and $x \in \bar{\Omega}$,

$$|\psi_i(x)| \leq max\left\{\|\beta_0\vec{\psi}\,(0)\|,\ \|\beta_1\vec{\psi}\,(1)\|,\quad \frac{1}{\alpha}\|L\vec{\psi}\|\right\}$$

Proof : Define the two functions,

$$\vec{\theta}^{\pm}(x) = max\left\{\|\beta_0\vec{\psi}\,(0)\|,\ \|\beta_1\vec{\psi}\,(1)\|,\quad \frac{1}{\alpha}\|L\vec{\psi}\|\right\}\vec{e} \pm \vec{\psi}(x),$$

where $\vec{e} = (1,..,1)^T$. Using the properties of $A(x)$, it is not hard to verify that $\beta_0\,\vec{\theta}^{\pm}(0) \geq \vec{0}, \beta_1\vec{\theta}^{\pm}\,(1) \geq \vec{0}$ and $L\vec{\theta}^{\pm}(x) \geq \vec{0}$ on Ω. It follows from Lemma 2.1 that $\vec{\theta}^{\pm}(x) \geq \vec{0}$ on $\bar{\Omega}$ as required.

Standard estimates of the solution (1.1),(1.2) and its derivatives are contained in the following lemma.

Lemma 2.3 Let $A(x)$ satisfy (1.5),(1.6) and let $\vec{u}$ be the solution of (1.1),(1.2). Then for each $i = 1, \dots, n$ and $x \in \bar{\Omega}$,

$$|u_i(x)| \leq C\|\vec{f}\|,\ |u'_i(x)| \leq C(\|\vec{u}\| + \|\vec{f}'\|),$$

$$|u''_i(x)| \leq C\varepsilon_i^{-\frac{1}{2}}(\|\vec{u}\| + \|\vec{f}'\|\ ,$$

$$\left|u_i^{(k)}(x)\right| \leq C\varepsilon_1^{-\frac{(k-3)}{2}}\varepsilon_i^{-1}\left(\|\vec{u}\| + \|\vec{f}\| + \varepsilon_1^{\frac{(k-3)}{2}}\left\|\vec{f}^{(k-2)}\right\|\right),$$

for $k = 3,4$.

Proof : The bound on $\vec{u}$ is an immediate consequence of Lemma 2.2.

Differentiating once the equation (1.1),

$$-E(\vec{u}')''(x) + A(x)\vec{u}'(x) = \vec{f}'(x) - A'(x)\vec{u}(x), \tag{2.1}$$

and from the boundary conditions,

$$\vec{u}'(0) = \vec{u}(0) - \vec{\phi}_0,\ \vec{u}'(1) = \vec{\phi}_1 - \vec{u}(1) \tag{2.2}$$

Replacing $\vec{u}'$ *by* $\vec{z}$ *in* (2.1), (2.2),

$$-E\vec{z}''(x) + A(x)\vec{z}(x) = \vec{h}(x) = \vec{f}'(x) - A'(x)\vec{u}(x), \tag{2.3}$$

with $\vec{z}(0) = \vec{u}(0) - \vec{\phi}_0, \vec{z}(1) = \vec{\phi}_1 - \vec{u}(1)$. (2.4)

This problem (2.3),(2.4) is similar to the problem in [8]. Now using the Lemma 2.2 in [8],

$|z_i(x)| \leq C(\|\vec{h}\|$. Therefore

$$|u'_i(x)| \leq C\left(\|\vec{u}\| + \|\vec{f}'\|\right).$$

Rewriting the differential equation (2.3) gives

$$\vec{z}'' = E^{-1}(A\vec{z} - \vec{h}) \tag{2.5}$$

and it is not hard to see that the bounds on z_i''follow from (2.5).

To bound $z_i'(x)$, for each i and x, consider an interval $\mathrm{N} = \left[a, a + \sqrt{\varepsilon_i}\right], a \geq 0$ such that $x \in N$. By the mean value theorem, for some $y \in N$, $a < y < a + \sqrt{\varepsilon_i}$,

$$z'_i(y) = \frac{z_i\left(a + \sqrt{\varepsilon_i}\right) - z_i(a)}{\sqrt{\varepsilon_i}}$$

which leads to,

$$|z_i'(y)| \leq C\varepsilon_i^{-\frac{1}{2}}\|\vec{z}\|. \tag{2.6}$$

Now, for any $x \in N$,

$$z_i'(x) = z'_i(y) + \int_y^x z_i''(s)ds.$$

By using (2.6), $|z_i'(x)| \leq C\varepsilon_i^{-\frac{1}{2}}(\|\vec{z}\| + \|\vec{h}\|)$. Hence,

$$|u_i''(x)| \leq C\varepsilon_i^{-\frac{1}{2}}\left(\|\vec{u}\| + \left\|\vec{f}'\right\|\right).$$

Differentiating the equation (1.1) once and using the bounds of u_i and $u_i{}'$ the bound of $u_i^{(3)}$ follow. Differentiating the equation (2.3) once and rearranging the equation, the bound of $z_i^{(3)}$ *or* $u_i^{(4)}$ follow.

Consider the Shishkin decomposition of the solution of $\vec{u}$of the BVP (1.1),(1.2) into smooth and singular components,

$$\vec{u} = \vec{v} + \vec{w} \tag{2.7}$$

Taking into consideration, the sublayers that appear for the components, the smooth component $\vec{v}$ is subjected to further decomposition

$$v_n = u_{0,n} + \varepsilon_n v_{n,n},$$
$$v_{n-1} = u_{0,n-1} + \varepsilon_n v_{n-1,n}^1,$$

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$$\qquad (2.8)$$

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$$v_1 = u_{0,1} + \varepsilon_n v_{1,n}^1,$$

as all the components have ε_n layers. Since components except u_n have ε_{n-1} sublayers, the components $v_{n-1}, \dots,$ v_1 takes the form,

$$v_{n-1} = u_{0,n-1} + \varepsilon_n\left(v_{n-1,n} + \varepsilon_{n-1}v_{n-1,n-1}\right),$$

$$v_{n-2} = u_{0,n-2} + \varepsilon_n\left(v_{n-2,n} + \varepsilon_{n-1}v_{n-2,n-1}^1\right),$$

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. $\qquad (2.9)$

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$$v_1 = u_{0,1} + \varepsilon_n\left(v_{1,n} + \varepsilon_{n-1}v_{1,n-1}^1\right).$$

Further, $u_{n-2}, u_{n-3}, \dots, u_2, u_1$ have ε_{n-2} sublayers and hence that leads to the decomposition,

$$v_{n-2} = u_{0,n-2} + \varepsilon_n\left(v_{n-2,n} + \varepsilon_{n-1}\left(v_{n-2,n-1} + \varepsilon_{n-2}v_{n-2,n-2}\right)\right)$$

$$v_{n-3} = u_{0,n-3} + \varepsilon_n\left(v_{n-3,n} + \varepsilon_{n-1}\left(v_{n-3,n-1} + \varepsilon_{n-2}v_{n-3,n-2}^1\right)\right)$$

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. $\qquad (2.10)$

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$$v_1 = u_{0,1} + \varepsilon_n\left(v_{1,n} + \varepsilon_{n-1}\left(v_{1,n-1} + \varepsilon_{n-2}v_{1,n-2}^1\right)\right)$$

Proceeding like this, it is not hard to see that

$$\overrightarrow{v}(x) = \overrightarrow{u_0}(x) + \vec{\gamma}(x) \qquad (2.11)$$

Where $\vec{\gamma}(x) = (\gamma_1, \gamma_2, \dots, \gamma_n)^T$,

$$\begin{pmatrix}\gamma_1\\ \gamma_2\\ \vdots\\ \gamma_n\end{pmatrix} = \begin{pmatrix}\varepsilon_1\varepsilon_2\dots\varepsilon_n & \varepsilon_2\varepsilon_3\dots\varepsilon_n & \dots & \varepsilon_n\\ 0 & \varepsilon_2\varepsilon_3\dots\varepsilon_n & \dots & \varepsilon_n\\ \vdots & \vdots & \dots & \vdots\\ 0 & 0 & \dots & \varepsilon_n\end{pmatrix}\begin{pmatrix}v_{1,1} & v_{1,2} & \dots & v_{1,n}\\ 0 & v_{2,2} & \dots & v_{2,n}\\ \vdots & \vdots & \dots & \vdots\\ 0 & 0 & \dots & v_{n,n}\end{pmatrix}^T \qquad (2.12)$$

That is,

$$\gamma_j = \vec{\varepsilon}_j^{\,j}\left(\vec{v}_j^{\,j}\right)^T \qquad (2.13)$$

$$\vec{\varepsilon}_j^{\,j} = (0,0,\dots,\varepsilon_j\varepsilon_{j+1}\dots\varepsilon_{n,}\varepsilon_{j+1}\varepsilon_{j+2}\dots\varepsilon_n, \varepsilon_{n-1}\varepsilon_n, \varepsilon_n)$$

$$\vec{v}_i^{\,i} = \left(0,0,\dots,v_{i,i}, v_{i,i+1},\dots,v_{i,n}\right).$$

Then using (2.7),(2.11) in (1.1),(1.2), it is found that the smooth component of the solution $\vec{u}$ satisfies

$$L\vec{v} = \vec{f}, \text{ on } \Omega, \qquad (2.14)$$

$$\beta_0\vec{v}(0) = \beta_0\vec{u}_0(0) + \beta_0\vec{\gamma}(0)$$

$$\beta_1\vec{v}(1) = \beta_1\vec{u}_0(1) + \beta_1\vec{\gamma}(1) \qquad (2.15)$$

From (2.9), (2.10) it is observed that the components $v_{i,j}$, $i = 1, \dots, n,\ j = i, i+1, \dots, n$ satisfy the following system of equations:

$$a_{11}v_{1,n} + a_{12}v_{2,n} + \dots + a_{1n}v_{n,n} = -\frac{\varepsilon_1}{\varepsilon_n}u_{0,1}''$$

$$a_{21}v_{1,n} + a_{22}v_{2,n} + \dots + a_{2n}v_{n,n} = -\frac{\varepsilon_2}{\varepsilon_n}u_{0,2}''$$

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. $\qquad (2.16)$

$$a_{n-11}v_{1,n} + a_{n-12}v_{2,n} + \dots + a_{n-1n}v_{n,n} = -\frac{\varepsilon_{n-1}}{\varepsilon_n}u_{0,n-1}''$$

$$\varepsilon_n v_{n,n} + a_{n1}v_{1,n} + a_{n2}v_{2,n} + \dots + a_{nn}v_{n,n} = -\frac{\varepsilon_n}{\varepsilon_n}u_{0,n}''$$

with

$$v_{n,n}(0) - v_{n,n}'(0) = 0, \quad v_{n,n}(1) + v_{n,n}'(1) = 0, \qquad (2.17)$$

$$a_{11}v_{1,n-1} + a_{12}v_{2,n-1} + \dots + a_{1n-1}v_{n-1,n-1} = -\frac{\varepsilon_1}{\varepsilon_{n-1}}v_{1,n}''$$

$$a_{21}v_{1,n-1} + a_{22}v_{2,n-1} + \dots + a_{2n-1}v_{n-1,n-1} = -\frac{\varepsilon_2}{\varepsilon_{n-1}}v_{2,n}''$$

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. $\qquad (2.18)$

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$$a_{n-21}v_{1,n-1} + a_{n-22}v_{2,n-1} + \dots + a_{n-2n-1}v_{n-1,n-1} = -\frac{\varepsilon_{n-2}}{\varepsilon_{n-1}}v_{n-2,n}''$$

$$\varepsilon_{n-1}v_{n-1,n-1}'' + a_{n-11}v_{1,n-1} + \dots + a_{n-1n-1}v_{n-1,n-1} = -\frac{\varepsilon_{n-1}}{\varepsilon_{n-1}}v_{n-1,n}''$$

with

$$v_{n-1,n-1}(0) - v_{n-1,n-1}'(0) = 0,$$

$$v_{n-1,n-1}(1) - v_{n-1,n-1}'(1) = 0 \qquad (2.19)$$

and so on. Lastly

$$a_{11}v_{1,2} + a_{12}v_{2,2} = -\frac{\varepsilon_1}{\varepsilon_2}v_{1,3}''$$

$$\varepsilon_2 v_{2,2}'' + a_{21}v_{1,2} + a_{22}v_{2,2} = -\frac{\varepsilon_2}{\varepsilon_2}v_{2,3}'' \qquad (2.20)$$

with $v_{2,2}(0) - v_{2,2}'(0) = 0, v_{2,2}(1) - v_{2,2}'(1) = 0$ $\qquad (2.21)$

and $\quad \varepsilon_1 v_{1,1}'' + a_{1,1}v_{1,1} = -v_{1,2}''$ $\qquad (2.22)$

$$v_{1,1}(0) - v_{1,1}'(0) = 0, \quad v_{1,1}(1) - v_{1,1}'(1) = 0. \qquad (2.23)$$

The singular component of the solution $\vec{u}$ satisfies

$$L\overrightarrow{w} = \vec{0}, \quad \text{on} \quad \Omega, \qquad (2.24)$$

$$\beta_0\overrightarrow{w}(0) = \beta_0\,(\vec{u} - \vec{v})(0), \quad \beta_1\overrightarrow{w}(1) = \beta_1\,(\vec{u} - \vec{v})(1) \qquad (2.25)$$

From the expression (2.16) – (2.23), using Lemma 2.3 for $\vec{v}$, it is found that for $i = 1, \dots, n,\ j = 1, \dots, n,\ i \le j,\ k =$ 0.1,2,3,4,

$$\left|v_{i,j}\right| \le C\left(1 + \prod_{r=j+1}^{n}\varepsilon_r^{-1/2}\prod_{r=j+2}^{n}\varepsilon_r^{-1/2}\right)$$

$$\left|v_{i,j}^{(k)}\right| \le C\left(1 + \varepsilon_j^{-\frac{k-1}{2}}\prod_{r=j+1}^{n}\varepsilon_r^{-1}\right). \qquad (2.26)$$

From (2.11),(2.13) and (2.26) the following bounds for $v_i, i = 1,2,\dots,n$ hold:

$$\left|v_i^{(k)}\right| \le C, \qquad k = 0,1,2,3$$

$$\left|v_i^{(4)}\right| \le C\left(1 + \varepsilon_i^{-1/2}\right), \qquad k = 4.$$

The layer functions $B_i^L, B_i^R, B_i, i = 1, \dots, n$, associated with the solution of $\vec{u}$, are defined on $\bar{\Omega}$ by

$$B_i^L(x) = e^{-x\sqrt{\frac{\alpha}{\varepsilon_i}}}, B_i^R(x) = B_i^L(1-x), B_i(x) = B_i^L(x) + B_i^R(x).$$

The following elementary properties of these layer functions, for all $1 \le i < j \le n$ and $0 \le x < y \le 1$, should be noted:

$B_i(x) = B_i(1-x), B_i^L(x) < B_j^L(x),\ B_i^L(x) > B_i^L(y), 0 <$ $B_i^L(x) \le 1, B_i^R(x) < B_j^R(x),\ B_i^R(x) < B_i^R(y), 0 <$

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$B_i^R(x) \leq 1$. $B_i(x)$ is monotone decreasing for increasing $x \in \left[0, \frac{1}{2}\right]$. $B_i(x)$ is monotone increasing for increasing $x \in \left[\frac{1}{2}, 1\right]$. $B_i(x) \leq 2B_i^L(x)$ for $x \in \left[0, \frac{1}{2}\right]$, $B_i(x) \leq 2B_i^R(x)$ for $x \in \left[\frac{1}{2}, 1\right]$, $B_i^L\left(2\frac{\sqrt{\varepsilon_i}}{\sqrt{\alpha}}\ln N\right) = N^{-2}$.

The interesting points $x_{i,j}^{(s)}$ are now defined.

Definition 2.1: For B_i^L, B_j^L, each $i, j, 1 \leq i \neq j \leq n$ and each $s, s > 0$, the points $x_{i,j}^{(s)}$ is defined by

$$\frac{B_i^L(x_{i,j}^{(s)})}{\varepsilon_i^s} = \frac{B_j^L(x_{i,j}^{(s)})}{\varepsilon_j^s} \qquad (2.27)$$

It is remarked that $\frac{B_i^R(1-x_{i,j}^{(s)})}{\varepsilon_i^s} = \frac{B_j^R(x_{i,j}^{(s)})}{\varepsilon_j^s}$ (2.28)

In the next lemma, the existence, uniqueness and ordering of the points $x_{i,j}^{(s)}$ are established. Sufficient conditions for them to lie in the domain $\overline{\Omega}$ are also provided.

Lemma 2.4: For all i, j such that $1 \leq i < j \leq n$ and such that $0 < s < 1$, the points $x_{i,j}^{(s)}$ exist, are uniquely defined and satisfy the following inequalities

$$\frac{B_i^L(x)}{\varepsilon_i^s} > \frac{B_j^L(x)}{\varepsilon_j^s}, x \in \left[0, x_{i,j}^{(s)}\right), \qquad \frac{B_i^L(x)}{\varepsilon_i^s} < \frac{B_j^L(x)}{\varepsilon_j^s}, x \in \left(x_{i,j}^{(s)}, 1\right]. \qquad (2.29)$$

In addition, the following ordering holds,

$$x_{ij}^{(s)} < x_{i+1j}^{(s)}, \text{ if } i+1 < j \text{ and } x_{ij}^{(s)} < x_{ij+1}^{(s)}, \text{ if } i < j. \qquad (2.30)$$

Also

$$x_{i,j}^{(s)} < 2s\frac{\sqrt{\varepsilon_j}}{\sqrt{\alpha}} \text{ and } x_{i,j}^{(s)} \in \left(0, \frac{1}{2}\right), \text{ if } i < j. \qquad (2.31)$$

Analogous results holds for B_i^R, B_j^R and the points $1 - x_{i,j}^{(s)}$.

Proof: The proof is as given in [8].

Bounds on the singular component $\vec{w}$ of $\vec{u}$ and its derivatives are contained in

Lemma 2.5 Let $A(x)$ satisfy (1.5),(1.6). Then there exist a constant C, such that, for each $i = 1, \ldots, n$ and $x \in \overline{\Omega}$,

$$|w_i(x)| \leq CB_n(x), |w_i'(x)| \leq CB_n(x),$$
$$|w_i^{(k)}(x)| \leq C\sum_{q=i}^{n}\frac{B_q(x)}{\varepsilon_q^{(k-1)/2}}, \text{ for } k = 2,3$$
$$|\varepsilon_i w_i^{(4)}(x)| \leq C\sum_{q=1}^{n}\frac{B_q(x)}{\sqrt{\varepsilon_q}}.$$

Proof : To derive the bound on $\vec{w}$, define $\vec{\theta}(x) = (\theta_1, \theta_2, \ldots, \theta_n)^T$, where $\theta_i^{\pm}(x) = CB_n(x) \pm w_i(x)$, for each $i = 1, \ldots, n$ and $x \in \overline{\Omega}$.

For a proper choice of C, $\beta_0\vec{\theta}^{\pm}(0) \geq \vec{0}$, $\beta_1\vec{\theta}^{\pm}(1) \geq \vec{0}$,. Also for $x \in \Omega$,

$$\left(L\vec{\theta}^{\pm}\right)_i(x) = -C\varepsilon_i\frac{\alpha}{\varepsilon_n}B_n(x) + C\sum_{j=1}^{n}a_{ij}(x)B_n(x) \pm 0$$

$$= C\left(\sum_{j=1}^{n}a_{ij}(x) - \alpha\left(\frac{\varepsilon_i}{\varepsilon_n}\right)\right)B_n(x) \geq 0, \qquad \text{as } -\frac{\varepsilon_i}{\varepsilon_n} > -1.$$

By Lemma 2.1, $\vec{\theta}^{\pm}(x) \geq \vec{0}$ on $\overline{\Omega}$ and it follows that,

$$|w_i(x)| \leq CB_n(x). \qquad (2.32)$$

The bounds on $w_i^{(k)}(x)$, $k = 1,2,3,4$ and $i = 1, \ldots, n$ are now derived by induction on n. For $n = 1$, the result follows from [9]. It is then assumed that the required bounds on w_i', w_i'', $w_i^{(3)}$ and $w_i^{(4)}$ hold for all systems up to order $n-1$. Define $\vec{w} = (w_1, \ldots w_{n-1})$, then $\vec{w}$ satisfies the system,

$$-\tilde{E}\vec{w}'' + \tilde{A}\vec{w} = \vec{g}, \qquad (2.33)$$

with

$$\beta_0\vec{w}(0) = \beta_0\vec{u}(0) - \beta_0\vec{u_0}(0),$$
$$\beta_1\vec{w}(1) = \beta_1\vec{u}(1) - \beta_1\vec{u_0}(1). \qquad (2.34)$$

Here, $\tilde{E}, \tilde{A}$ are the matrices obtained by deleting the last row and last column from E, A respectively and the components of $\vec{g}$ are $g_i = -a_{i,n}w_n$, for $1 \leq i \leq n-1$ and $\vec{v} = \vec{u_0} + \vec{\gamma}$ the corresponding components decomposition of $\vec{v}$ is similar to (2.11) of $\vec{v}$. Now decompose $\vec{w}$ into smooth and singular components to get, $\vec{w} = \vec{p} + \vec{r}$, where $L\vec{p} = \vec{g}$, $\beta_0\vec{p}(0) = \beta_0\vec{u_0}(0) + \beta_0\vec{\gamma}(0)$, $\beta_1\vec{p}(1) = \beta_1\vec{u_0}(1) + \beta_1\vec{\gamma}(0)$ and $L\vec{r} = \vec{0}$, $\beta_0\vec{r}(0) = \beta_0\vec{w}(0) - \beta_0\vec{p}(0)$, $\beta_1\vec{r}(1) = \beta_1\vec{w}(1) - \beta_1\vec{p}(1)$.

By induction, the bounds on the derivatives of $\vec{w}$ hold. That is for, $i = 1, \ldots, n-1$,

$$|w_i'(x)| \leq C\sum_{q=i}^{n-1}B_q(x), \quad |w_i''(x)| \leq C\sum_{q=i}^{n-1}\frac{B_q(x)}{\sqrt{\varepsilon_q}},$$
$$\left|w_i^{(3)}(x)\right| \leq C\sum_{q=1}^{n-1}\frac{B_q(x)}{\varepsilon_q}, \quad \left|\varepsilon_i w_i^{(4)}(x)\right| \leq C\sum_{q=1}^{n-1}\frac{B_q(x)}{\sqrt{\varepsilon_q}}. \qquad (2.35)$$

Consider the n^{th} equation of the system satisfied by w_n,

$$-\varepsilon_n w_n''(x) + \sum_{j=1}^{n-1}a_{nj}(x)\widetilde{w_j}(x) + a_{nn}(x)w_n(x) = 0 \qquad (2.36)$$

Differentiating (2.36) once,

$$-\varepsilon_n w_n^{(3)}(x) + \sum_{j=1}^{n-1}a_{nj}(x)\widetilde{w_j}'(x) + a_{nn}(x)w_n'(x) = -\sum_{j=1}^{n-1}a'_{nj}(x)\widetilde{w_j}(x) - a'_{nn}w_n(x) \qquad (2.37)$$

From the boundary conditions,

$$w'_n(0) = w_n(0) - \left[\beta_0(\vec{u} - \vec{v})\right]_n(0), \quad w'_n(1) = \left[\beta_1(\vec{u} - \vec{v})\right]_n(1) - w_n(1) \qquad (2.38)$$

Replacing $\widetilde{w_j}'$ by $\widetilde{z_j}$, $j = 1, \ldots, n-1$ and w'_n by $\widetilde{z_n}$ in (2.37) and (2.38),

$$-\varepsilon_n(\widetilde{z_n}''(x) + \sum_{j=1}^{n}a_{nj}(x)\widetilde{z_j}(x) = -\sum_{j=1}^{n-1}a'_{nj}(x)\widetilde{w_j}(x) - a'_{nn}w_n(x), \qquad (2.39)$$

with $\widetilde{z_n}(0) = w_n(0) - \left[\beta_0(\vec{u} - \vec{v})\right]_n(0)$,

$$\widetilde{z_n}(1) = \left[\beta_1(\vec{u} - \vec{v})\right]_n(1) - w_n(1) \qquad (2.40)$$

This problem (2.39), (2.40) is similar to the problem in [8]. Now, using Lemma 2.2 in [8], the bound on $\widetilde{z_n}$ is determined. Thus,

$$|w_n'(x)| \leq CB_n(x).$$

By using (2.32) in (2.39),

$$\left|(\widetilde{z_n})''(x)\right| \leq \frac{C}{\varepsilon_n}B_n(x). \qquad (2.41)$$

By using the mean value theorem,

$$|(\widetilde{z_n})'(x)| \leq \frac{C}{\sqrt{\varepsilon_n}}B_n(x). \qquad (2.42)$$

Therefore, $$|w_n''(x)| \leq \frac{C}{\sqrt{\varepsilon_n}}B_n(x). \qquad (2.43)$$

Now, differentiating the equation satisfied by w_n and rearranging gives,

 DOI : 10.18231/2454-9150.2018.0752

$$\varepsilon_n w_n^{(3)}(x) = \sum_{q=1}^{n-1} a_{nq}(x)\widetilde{w_q}'(x) + a_{nn}(x)w_n'(x) + \sum_{q=1}^{n-1} a'_{nq}(x)\widetilde{w_q}(x) + a'_{nn}(x)w_n(x).$$

The bounds on w_n and (2.35), then gives

$$\left|w_n^{(3)}(x)\right| \le C\sum_{q=1}^{n} \varepsilon_q^{-1}B_q(x).$$

Differentiating (2.39) once and rearranging yields,

$$\left|\varepsilon_n(\widetilde{z_n})^{(3)}(x)\right| \le C\sum_{q=1}^{n} \varepsilon_q^{-\frac{1}{2}}B_q(x).$$

$$ie., \left|\,\varepsilon_n w_n^{(4)}(x)\right| \le C\sum_{q=1}^{n} \varepsilon_q^{-1/2}B_q(x).$$

Using the bounds on $w_n(x), w_n'(x), w_n''(x), w_n^{(3)}(x)$ and $w_n^{(4)}(x)$ it is seen that the function $\vec{g}$ in (2.33) and its derivatives $\vec{g}\,'(x), \vec{g}\,''(x), \vec{g}^{(3)}(x), \vec{g}^{(4)}(x)$ are bounded by $CB_n(x), CB_n(x), C\frac{B_n(x)}{\sqrt{\varepsilon_n}}, C\sum_{q=1}^{n}\frac{B_q(x)}{\varepsilon_q}$ and $C\varepsilon_n^{-1}\sum_{q=1}^{n}\frac{B_q(x)}{\sqrt{\varepsilon_q}}$ respectively.

By induction, the following bounds for $\vec{p}$ and $\vec{r}$ hold for $i = 1,\dots,n-1$,

$$|p_i'(x)| \le C, \quad |r_i'(x)| \le C\big(B_i(x) + \dots + B_{n-1}(x)\big),$$

$$|p_i''(x)| \le C, \quad |r_i''(x)| \le C\left(\frac{B_i(x)}{\sqrt{\varepsilon_i}} + \dots + \frac{B_{n-1}(x)}{\sqrt{\varepsilon_{n-1}}}\right)$$

$$\left|p_i^{(3)}(x)\right| \le C, \quad \left|r_i^{3}(x)\right| \le C\left(\frac{B_i(x)}{\varepsilon_1} + \dots + \frac{B_{n-1}(x)}{\varepsilon_{n-1}}\right),$$

$$\left|p_i^{(4)}(x)\right| \le C\left(1+\varepsilon_i^{-1/2}\right), \ \left|\varepsilon_i r_i^{4}(x)\right| \le C\left(\frac{B_1(x)}{\sqrt{\varepsilon_1}} + \dots + \frac{B_{n-1}(x)}{\sqrt{\varepsilon_{n-1}}}\right),$$

Introducing the functions, $\vec{\psi}^{\pm}(x) = CB_n(x)\vec{e} \pm \vec{p}(x)$, then clearly $\beta_0\vec{\psi}^{\pm}(0) = CB_n(0)\vec{e} \pm \beta_0\vec{p}(0) \ge \vec{0}$, $\beta_1\vec{\psi}^{\pm}(1) = CB_n(1)\vec{e} \pm \beta_1\vec{p}(1) \ge \vec{0}$, and

$$\left(L\vec{\psi}^{\pm}\right)_i(x) = C\left(\sum_{j=1}^{n} a_{ij}(x) - \alpha\left(\frac{\varepsilon_i}{\varepsilon_n}\right)\right)B_n(x) \pm L\vec{p} \ge 0, \quad \text{as } -\frac{\varepsilon_i}{\varepsilon_n} \ge -1.$$

Applying Lemma 2.1, it follows that $\|\vec{p}(x)\| \le CB_n(x)$.

Defining barrier functions, $\vec{\theta}^{\pm}(x) = C\varepsilon_n^{-(k-1)/2}B_n(x)\vec{e} \pm \vec{p}^{k}(x), \quad k = 1,2$ and using Lemma 2.1 for the problem satisfied by $\vec{p}$, the bounds required for $\vec{p}'$ and $\vec{p}''$ are derived.

The bounds for $\vec{p}^{(k)}$, $k = 3,4$ can be derived by differentiating the defining equation of $\vec{p}$ and using the bounds of $\vec{p}^{(k)}$, $k = 1,2$.

Combining the bounds for the derivatives of p_i and r_i, it follows that

$$\left|\widetilde{w}_i'(x)\right| \le C\sum_{q=i}^{n-1} B_q(x), \quad \left|\widetilde{w}_i''\right| \le C\sum_{q=i}^{n-1}\frac{B_q(x)}{\sqrt{\varepsilon_q}},$$

$$\left|\widetilde{w}_i^{(3)}\right| \le C\sum_{q=i}^{n-1}\frac{B_q(x)}{\varepsilon_q}, \quad \left|\varepsilon_i\widetilde{w}_i^{(4)}\right| \le C\sum_{q=1}^{n-1}\frac{B_q(x)}{\sqrt{\varepsilon_q}}.$$

Using the above bounds along with the bounds of w_n, the proof of the lemma for the system of n equations is completed.

III. THE SHISHKIN MESH

A piecewise uniform Shishkin mesh on $\bar{\Omega}$ with N mesh-intervals is now constructed. Let $\Omega^N = \{x_j\}_{j=1}^{N-1}$ and $\bar{\Omega}^N = \{x_j\}_{j=0}^{N}$. The mesh $\bar{\Omega}^N$ is a piecewise–uniform mesh on $\bar{\Omega} = [0,1]$ obtained by dividing $[0,1]$ into $2n+1$ mesh-intervals given by,

$$[0,\tau_1] \cup \dots \cup (\tau_{n-1},\tau_n] \cup (\tau_n, 1-\tau_n] \cup (1-\tau_n, \tau_{n-1}] \cup \dots \cup (1-\tau_1, 1].$$

The n parameters τ_r, which determine the points separating the uniform meshes, are defined by ,

$$\tau_n = min\left\{\frac{1}{4}, 2\frac{\sqrt{\varepsilon_n}}{\sqrt{\alpha}}\ln N\right\} \tag{3.1}$$

and , for $r = n-1,\dots,1$,

$$\tau_r = min\left\{\frac{r\tau_{r+1}}{r+1}, 2\frac{\sqrt{\varepsilon_r}}{\sqrt{\alpha}}\ln N\right\}. \tag{3.2}$$

Also, $\tau_0 = 0$, $\tau_{n+1} = 1/2$.

Clearly, $0 < \tau_1 < \dots < \tau_n \le \frac{1}{4}$, $\frac{3}{4} \le 1-\tau_n < \dots < 1-\tau_1 < 1$.

Then, on the subinterval $(\tau_n, 1-\tau_n]$, a uniform mesh with $\frac{N}{2}$ mesh-intervals is placed and on each of the mesh-intervals $(\tau_r, \tau_{r+1}]$ and $(1-\tau_{r+1}, 1-\tau_r]$, $r = 0,1,\dots,n-1$, a uniform mesh of $\frac{N}{4n}$ mesh-intervals is placed. In practice, it is convenient to take

$$N = 4nk, \quad k \ge 3 \tag{3.3}$$

where n is the number of distinct singular perturbation parameters involved in (1.1). This construction leads to a class of 2^n piecewise uniform meshes $\bar{\Omega}^N$.

In particular, when all the parameters $\tau_r, r = 1,\dots,n$ are with the left choice, the Shishkin mesh $\bar{\Omega}^N$ becomes the classical uniform mesh with the transition parameters $\tau_r = \frac{r}{4n}, r = 1,\dots,n$ with step size N^{-1} .

The Shishkin mesh suggested here has the following features: (i) when all the transition parameters have the left choice, it is the classical uniform mesh and (ii) it is coarse in the outer region and becomes finer and finer towards the left and right boundaries.

From the above construction it is clear that the transition points $\{\tau_r, 1-\tau_r\}_{r=1}^{n}$ are the only points at which the mesh size can change and that it does not necessarily change at each of these points. The following notations are introduced: $h_{j+1} = x_{j+1} - x_j$, $h_j = x_j - x_{j-1}$, and if $x_j = \tau_r$, then $h_r^+ = x_{j+1} - x_j$, $h_r^- = x_j - x_{j-1}$, $J = \{\tau_r, 1-\tau_r : h_r^+ \ne h_r^-\}$. In general, for each point x_j in the mesh interval $(\tau_{r-1}, \tau_r]$,

$$x_j - x_{j-1} = 4\mathrm{n}\mathrm{N}^{-1}(\tau_r - \tau_{r-1}). \tag{3.4}$$

Also, for $x_j \in \left(\tau_{n.}\frac{1}{2}\right]$, $x_j - x_{j-1} = 2\mathrm{N}^{-1}(1-2\tau_n)$ and for $x_j \in (0,\tau_1]$, $x_j - x_{j-1} = 4\mathrm{n}\mathrm{N}^{-1}\tau_1$. Thus, for $1 \le r \le n-1$, the change in the step size at the point $x_j = \tau_r$ is,

$$h_r^+ - h_r^- = 4\mathrm{n}\mathrm{N}^{-1}\left(\frac{(\mathrm{r}+1)}{\mathrm{r}}d_r - d_{r-1}\right), \tag{3.5}$$

where,

$$d_r = \frac{r\tau_{r+1}}{r+1} - \tau_r \tag{3.6}$$

with the convention $d_0 = 0$. Notice that, $d_r \ge 0$, Ω^N is the classical uniform mesh when $d_r = 0$ for all $r = 1,\dots,n$ and from (44) and (45), that

$$\tau_r \le C\sqrt{\varepsilon_r}\ln N. \tag{3.7}$$

It follows from (47) and (50) that, for $1 \le r \le n-1$,

$$h_r^+ + h_r^- \le C\sqrt{\varepsilon_{r+1}}\,\mathrm{N}^{-1}\ln N. \tag{3.8}$$

Also, $\tau_r = \frac{r}{s}\tau_s$, when $d_r = \dots = d_s = 0, 1 \le r \le s \le n$.

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The results in the following lemma are used later.

Lemma 3.1 Assume that $d_r > 0$ for some r, $1 \le r \le n$. Then the following inequalities hold,

$$B_r^L(1-\tau_r) \le B_r^L(\tau_r) = N^{-2}, \quad (3.9)$$

$$x_{r-1,r}^{(s)} \le \tau_r - h_r^{-} \text{ for } 0 < s \le 1, \quad (3.10)$$

$$B_q^L(\tau_r - h_r^{-}) \le C B_q^L(\tau_r), \text{ for } 1 \le r \le q \le n \quad (3.11)$$

$$\frac{B_q^L(\tau_r)}{\sqrt{\varepsilon_q}} \le C \frac{1}{\sqrt{\varepsilon_r} lnN} \text{ for } 1 \le q \le n. \quad (3.12)$$

Analogous results hold for B_r^R

Proof: The proof is as given in [8].

IV. THE DISCRETE PROBLEM

In this section, a classical finite difference operator with an appropriate Shishkin mesh is used to construct a numerical method for the problem (1.1),(1.2) which is shown later to be essentially first order parameter-uniform convergent.

The discrete two-point boundary value problem is now defined by the finite difference scheme on the Shishkin mesh defined in the previous section.

$$-E\delta^2 \vec{U}(x_j) + A(x)\vec{U}(x_j) = \vec{f}(x_j),\ 1 \le j \le N-1 \quad (4.1)$$

with

$$\vec{U}(0) - D^{+}\vec{U}(0) = \vec{\phi}_0, \qquad \vec{U}(1) + D^{-}\vec{U}(1) = \vec{\phi}_1 \ . \quad (4.2)$$

The problem (4.1), (4.2) can also be written in the operator form

$$L^N\vec{U} = \vec{f} \quad \text{on} \quad \Omega^N,$$
$$\beta_0{}^N\vec{U}(0) = \vec{\phi}_0, \qquad \beta_1{}^N\vec{U}(1) = \vec{\phi}_1,$$

where

$$L^N = -E\delta^2 + A, \qquad \beta_0{}^N = I - D^{+}I,$$
$$\beta_1{}^N\vec{U}(1) = I + D^{-}I,$$

and $D^{+}, D^{-}$and δ^2 are the difference operators

$$D^{+}\vec{U}(x_j) = \frac{\vec{U}(x_{j+1}) - \vec{U}(x_j)}{x_{j+1} - x_j}, \quad D^{-}\vec{U}(x_j) = \frac{\vec{U}(x_j) - \vec{U}(x_{j-1})}{x_j - x_{j-1}}$$

and

$$\delta^2\vec{U}(x_j) = \frac{D^{+}\vec{U}(x_j) - D^{-}\vec{U}(x_j)}{(x_{j+1} - x_{j-1})/2}, \qquad 1 \le j \le N-1$$

For any function $\vec{Z} = (Z_1, Z_2, \ldots, Z_n)^T$defined on the Shishkin mesh $\bar{\Omega}^N$, the following norm $\|Z\| = max_{1 \le i \le n} max_{0 \le j \le N} |Z_i(x_j)|$ is introduced.

The following discrete results are analogous to those for the continuous case.

Lemma 4.1 Let $A(x)$satisfy (5),(6). Let $\vec{\Psi}$ be any vector-valued mesh function, such that $\beta_0{}^N\vec{\Psi}(0) \ge \vec{0}$ $\beta_1{}^N\vec{\Psi}(1) \ge \vec{0}$. Then $L^N\vec{\Psi} \ge \vec{0}$ on Ω^N implies that $\vec{\Psi} \ge \vec{0}$ on $\bar{\Omega}^N$.

Proof: Let i^*, j^* be such that $\Psi_{i^*}(x_j) = min_{1 \le i \le n} min_{0 \le j \le N} \Psi_i(x_j)$ and assume that the lemma is false. Then, $\Psi_{i^*}(x_{j^*}) < 0$. If, $x_{j^*} = 0$, then $(\beta_0{}^N\vec{\Psi})_{i^*}(0) = \Psi_{i^*}(0) - D^{+}\Psi_{i^*}(0) < 0$, a contradiction. Therefore, $x_{j^*} \neq 0$ and for the same reason $x_{j^*} \neq 1$. $\Psi_{i^*}(x_{j^*}) - \Psi_{i^*}(x_{j^*-1}) \le 0$, $\Psi_{i^*}(x_{j^*+1}) - \Psi_{i^*}(x_{j^*}) \ge 0$. Also, $\delta^2\Psi_{i^*}(x_{j^*}) > 0$. It follows that,

$$(L^N\vec{\Psi})_{i^*}(x_{j^*}) = -\varepsilon_{i^*}\delta^2\,\Psi_{i^*}(x_{j^*}) + a_{i^*i^*}(x_{j^*})\Psi_{i^*}(x_{j^*}) + \sum_{k=1, k\neq i^*}^{n} a_{i^*k}(x_{j^*})\,\Psi_k(x_{j^*}) < 0,$$

which is a contradiction. Hence the result.

An immediate consequence of this is the following discrete stability result.

Lemma 4.2 Let $A(x)$ satisfy (1.5),(1.6). Let $\vec{\Psi}$ be any vector-valued mesh function on $\bar{\Omega}^N$, then for each , $i = 1, \ldots, n$,

$$|\Psi_i(x_j)| = max\left\{\|\beta_0{}^N\vec{\Psi}\ (0)\|,\ \|\ \beta_1{}^N\vec{\Psi}\ (1)\|,\ \frac{1}{\alpha}\|L^N\vec{\Psi}\|\right\}, \qquad 0 \le j \le N$$

Proof : Define the two mesh functions,

$$\vec{\Theta}^{\pm}(x_j) = max\left\{\|\beta_0{}^N\vec{\Psi}\ (0)\|,\ \|\ \beta_1{}^N\vec{\Psi}\ (1)\|,\ \frac{1}{\alpha}\|L^N\vec{\Psi}\|\right\}\vec{e} \pm \vec{\Psi}(x_j).$$

Using the properties of $A(x)$, it is not hard to verify that $\beta_0{}^N\vec{\Theta}^{\pm}(0) \ge \vec{0}$, $\beta_1{}^N\vec{\Theta}^{\pm}(1) \ge \vec{0}$ and $L^N\vec{\Theta}^{\pm} \ge \vec{0}$ on Ω^N. It follows from Lemma 7 that $\vec{\Theta}^{\pm} \ge \vec{0}$ on $\bar{\Omega}^N$.

The following comparison principle will be used in the proof of the error estimate.

Lemma 4.3 Assume that, for each $i = 1, \ldots, n$, the vector-valued mesh functions $\vec{\Phi}$ and $\vec{Z}$ satisfy $\left|(\beta_0{}^N\vec{Z})_i(0)\right| \le (\beta_0{}^N\vec{\Phi})_i(0)$, $\left|(\beta_1{}^N\vec{Z})_i(1)\right| \le (\beta_1{}^N\vec{\Phi})_i(1)$ and $\left|(L^N\vec{Z})_i\right| \le (L^N\vec{\Phi})_i$ on Ω^N. Then, for each $i = 1, \ldots, n$, $|Z_i| \le \Phi_i$ on $\bar{\Omega}^N$.

Proof: Define the two mesh functions, $\vec{\Psi}^{\pm}$ by $\vec{\Psi}^{\pm} = \vec{\Phi} \pm \vec{Z}$ Then, for each $i = 1, \ldots, n$, $\Psi_i{}^{\pm}$ satisfies $(\beta_0{}^N\vec{\Psi}^{\pm})_i(0) \ge 0$, $(\beta_1{}^N\vec{\Psi}^{\pm})_i(1) \ge 0$ and $(L^N\vec{\Psi}^{\pm})_i$ on Ω^N. The required result follows from Lemma 7.

V. THE LOCAL TRUNCATION ERROR

From Lemma 4.2, it is seen that in order to bound the error, $\vec{U} - \vec{u}$ it suffices to bound $L^N(\vec{U} - \vec{u})$. Notice that, for $x_j \in \Omega^N$,

$$L^N(\vec{U} - \vec{u}) = L^N\vec{U} - L^N\vec{u} = \vec{f} - L^N\vec{u} = L\vec{u} - L^N\vec{u}$$
$$= (L - L^N)\vec{u} = -E(\delta^2 - D^2)\vec{u}$$

which is the local truncation of the second derivative.

Let $\vec{V}, \vec{W}$ be the discrete analogous of $\vec{v}, \vec{w}$, respectively. Then,

$$L^N\vec{V} = \vec{f} \quad \text{on} \quad \Omega^N,$$
$$\beta_0{}^N\vec{V}(0) = \beta_0\vec{v}(0),\ \beta_1{}^N\vec{V}(1) = \beta_1\vec{v}(1), \quad (5.1)$$

and

$$L^N\vec{W} = \vec{f} \quad \text{on} \quad \Omega^N,$$
$$\beta_0{}^N\vec{W}(0) = \beta_0\vec{w}(0), \beta_1{}^N\vec{W}(1) = \beta_1\vec{w}(1) \quad (5.2)$$

where $\vec{v}$ and $\vec{w}$ are the solutions of (2.14),(2.15) and (2.24),(2.25) respectively.

Further,

$$\beta_0{}^N(\vec{V} - \vec{v})(0) = (D - D^{+})\vec{v}(0),$$
$$\beta_1{}^N(\vec{V} - \vec{v})(1) = (D^{-} - D)\vec{v}(1)$$

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$$\beta_0{}^N(\vec{W}-\vec{w})(0) = (D-D^+)\vec{w}(0),$$
$$\beta_1{}^N(\vec{W}-\vec{w})(1) = (D^- - D)\vec{w}(1)$$
$$L^N(\vec{V}-\vec{v})(x_j) = -E(D^2-\delta^2)\vec{v}(x_j)$$
$$L^N(\vec{W}-\vec{w})(x_j) = -E(D^2-\delta^2)\vec{w}(x_j)$$

and so, for each $i = 1,\dots,n$,

$$\left|\left(\beta_0{}^N(\vec{V}-\vec{v})\right)_i(0)\right| = |(D-D^+)|v_i(0),$$
$$\left|\left(\beta_1{}^N(\vec{V}-\vec{v})\right)_i(1)\right| = |(D^- - D)|v_i(1),$$
$$\left|\left(\beta_0{}^N(\vec{W}-\vec{w})\right)_i(0)\right| = |(D-D^+)|w_i(0),$$
$$\left|\left(\beta_1{}^N(\vec{W}-\vec{w})\right)_i(1)\right| = |(D^- - D)|w_i(1),$$

$$\left|\left(L^N(\vec{V}-\vec{v})\right)_i(x_j)\right| \le |\varepsilon_i(D^2-\delta^2)v_i(x_j)| \quad (5.3)$$

$$\left|\left(L^N(\vec{W}-\vec{w})\right)_i(x_j)\right| \le |\varepsilon_i(D^2-\delta^2)w_i(x_j)| \quad (5.4)$$

Therefore, the local truncation error of the smooth and singular components can be treated separately. In view of this, it is to be noted that, for any smooth function ψ and for each $x_j \in \Omega^N$, the following expressions may be used to estimate the local truncation error.

$$|(D-D^-)\psi(x_j)| \le C(x_j - x_{j-1})max_{s\in[x_{j-1},x_j]}|\psi^{(2)}(s)|. \quad (5.5)$$

$$|(D-D^+)\psi(x_j)| \le C(x_{j+1} - x_j)max_{s\in[x_j,x_{j+1}]}|\psi^{(2)}(s)|. \quad (5.6)$$

$$|(\delta^2-D^2)\psi(x_j)| \le Cmax_{s\in I_j}|\psi^{(2)}(s)|. \quad (5.7)$$

$$|(\delta^2-D^2)\psi(x_j)| \le C(x_{j+1} - x_{j-1})max_{s\in I_j}|\psi^{(3)}(s)|. \quad (5.8)$$

Here, $I_j = [x_{j-1}, x_{j+1}]$.

VI. ERROR ESTIMATE

The proof of the theorem on the error estimate is split into two parts. First, a theorem concerning the error in the smooth component is established. Then the error in the singular component is estimated.

Define the barrier function $\vec{\Phi}$ by,

$$\vec{\Phi}(x_j) = C\left[(r+1)(N^{-1}\ln N) + (N^{-1}\ln N)\sum_{\{r:\tau_r\in J\}}\frac{\tau_r}{\sqrt{\varepsilon_i}}\theta_r(x_j)\right](\vec{e}) \quad (6.1)$$

where C is any sufficiently large constant and θ_r is a piecewise linear polynomial on $\bar{\Omega}$, defined by,

$$\theta_r(x) = \begin{cases} \dfrac{x}{\tau_r}, & 0 \le x \le \tau_r \\ 1, & \tau_r < x < 1-\tau_r \\ \dfrac{1-x}{\tau_r}, & 1-\tau_r \le x \le 1. \end{cases}$$

Also note that,

$$(L^N\theta_r\vec{e})_i(x_j) \ge \begin{cases} \alpha\theta_r(x_j), & if\ x_j \notin J \\ \alpha + \dfrac{2\varepsilon_i}{\tau_r(h_r^+ + h_r^-)}, & if\ x_j \in J. \end{cases} \quad (6.2)$$

Then, on Ω^N, $\vec{\Phi}$ satisfy,

$$0 \le \Phi_i(x_j) \le CN^{-1}\ln N,\ \ 1 \le i \le n. \quad (6.3)$$

Also,

$$(\beta_0{}^N\vec{\Phi})_i(0) \ge CN^{-1}\ln N,\ \ (\beta_1{}^N\vec{\Phi})_i(1) \ge CN^{-1}\ln N. \quad (6.4)$$

For $x_j \notin J$,

$$(L^N\vec{\Phi})_i(x_j) \ge CN^{-1}\ln N, \quad (6.5)$$

and, $x_j \in J$, using (3.7),(3.8) and (6.2),

$$(L^N\vec{\Phi})_i(x_j) \ge CN^{-1}\ln N. \quad (6.6)$$

The following theorem gives the estimate of error in the singular component.

Theorem 6.1 Let $A(x)$ satisfy (1.5),(1.6). Let $\vec{v}$ denote the smooth component of the solution of the problem (1.1),(1.2) and $\vec{V}$ be the smooth component of the solution of the problem (4.1),(4,2). Then

$$||\vec{V}-\vec{v}\,|| \le CN^{-1}\ln N. \quad (6.7)$$

Proof : From the expression (5.6) and (5.5),

$$\left|\left(\beta_0{}^N(\vec{V}-\vec{v})\right)_i(0)\right| \le C(x_1 - x_0)max_{s\in[x_0,x_1]}|v_i''(s)| \le CN^{-1} \quad (6.8)$$

$$\left|\left(\beta_1{}^N(\vec{V}-\vec{v})\right)_i(1)\right| \le C\sqrt{\varepsilon_i}(x_N - x_{N-1})max_{s\in[x_{N-1},x_N]}|v_i''(s)| \le CN^{-1}. \quad (6.9)$$

Thus from (6.4),(6.8) and (6.9),

$$\left|\left(\beta_0{}^N(\vec{V}-\vec{v})\right)_i(0)\right| \le (\beta_0{}^N\vec{\Phi})_i(0),$$
$$\left|\left(\beta_1{}^N(\vec{V}-\vec{v})\right)_i(1)\right| \le (\beta_1{}^N\vec{\Phi})_i(1). \quad (6.10)$$

For each mesh point x_j, there are two possibilities: either $x_j \notin J$ or $x_j \in J$.

For $x_j \notin J$, using the bounds of the derivatives of $\vec{v}$ and the expression (65),

$$\left|\left(L^N(\vec{V}-\vec{v})\right)_i(x_j)\right| \le CN^{-1}. \quad (6.11)$$

On the other hand , if $x_j \in J$, then $x_j \in \{\tau_r, 1-\tau_r\}$, for some r, $1 \le r \le n$.

Consider the case $x_j = \tau_r$ and for $x_j = 1-\tau_r$, the proof is analogous.

If $x_j = \tau_r \in J$, using the bounds of the derivatives of $\vec{v}$ and the expression (65),

$$\left|\left(L^N(\vec{V}-\vec{v})\right)_i(x_j)\right| \le CN^{-1}\ln N. \quad (6.12)$$

From (6.10),(6.11),(6.12) and Lemma 4.3, the required result is obtained.

In order to estimate the error in the singular component of the solution $\vec{u}$, the following lemmas are required.

Lemma 6.1 Assume that $x_j \notin J$. Let $A(x)$ satisfy (1.5),(1.6). Then on Ω^N, for each $1 \le i \le n$,

$$\left|\left(L^N(\vec{W}-\vec{w})\right)_i(x_j)\right| \le Cx_{j+1} - x_{j-1.} \quad (6.13)$$

The following decomposition of the singular components w_i are used in the next lemma.

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$$w_i = \sum_{m=1}^{r+1} w_{i,m} \ , \qquad (6.14)$$

where the components $w_{i,m}$ are defined by

$$w_{i,r+1} = \begin{cases} p_i^{(s)} \ on \ [0, x_{r,r+1}^{(s)}) \\ w_i \ on \ [x_{r,r+1}^{(s)}, 1 - x_{r,r+1}^{(s)}] \\ q_i^{(s)} \ on \ (1 - x_{r,r+1}^{(s)}, 1] \end{cases}$$

where

$$p_i^{(s)}(x) = \begin{cases} \sum_{k=0}^{3} w_i^{(k)} x_{r,r+1}^{(s)} \dfrac{\left(x - x_{r,r+1}^{(s)}\right)^k}{k!}, & s = 1 \\ \sum_{k=0}^{4} w_i^{(k)} x_{r,r+1}^{(s)} \dfrac{(x - x_{r,r+1}^{(s)})^k}{k!} \ , & s = 1/2, \end{cases}$$

$$q_i^{(s)}(x) = \begin{cases} \sum_{k=0}^{3} w_i^{(k)} (1 - x_{r,r+1}^{(s)}) \dfrac{\left(x - (1 - x_{r,r+1}^{(s)})\right)^k}{k!}, & s = 1 \\ \sum_{k=0}^{4} w_i^{(k)} (1 - x_{r,r+1}^{(s)}) \dfrac{(x - (1 - x_{r,r+1}^{(s)}))^k}{k!} \ , & s = 1/2, \end{cases}$$

and for each $m, \ r \geq m \geq 2$,

$$w_{i,m} = \begin{cases} p_i^{(s)} \ on \ [0, x_{m-1,m}^{(s)}) \\ w_i - \sum_{k=m+1}^{r+1} w_{i,k} \ on \ [x_{m-1,m}^{(s)}, 1 - x_{m-1,m}^{(s)}] \\ q_i^{(s)} \ on \ (1 - x_{m-1,m}^{(s)}, 1] \end{cases}$$

where

$$p_i^{(s)}(x) = \begin{cases} \sum_{k=0}^{3} w_i^{(k)} x_{m,m+1}^{(s)} \dfrac{\left(x - x_{m,m+1}^{(s)}\right)^k}{k!}, & s = 1 \\ \sum_{k=0}^{4} w_i^{(k)} x_{m,m+1}^{(s)} \dfrac{(x - x_{m,m+1}^{(s)})^k}{k!} \ , & s = 1/2, \end{cases}$$

$$q_i^{(s)}(x) = \begin{cases} \sum_{k=0}^{3} w_i^{(k)} (1 - x_{m,m+1}^{(s)}) \dfrac{\left(x - (1 - x_{m,m+1}^{(s)})\right)^k}{k!}, & s = 1 \\ \sum_{k=0}^{4} w_i^{(k)} (1 - x_{m,m+1}^{(s)}) \dfrac{(x - (1 - x_{m,m+1}^{(s)}))^k}{k!} \ , & s = 1/2, \end{cases}$$

and

$$w_{i,1} = \ w_i - \sum_{k=2}^{r+1} w_{i,k} \quad on \ [0,1].$$

Notice that the decomposition (6.14) depends on the choice of the polynomials $p_i^{(s)}, q_i^{(s)}$ and the definition of $x_{i,j}^{(s)}$ given in (2.27).

The following lemma provides estimates of the derivatives of the components (6.14).

Lemma 6.2 Assume that $d_r > 0$ for some $r, \ 1 \leq r \leq n$. Let $A(x)$ satisfy (1.5),(1.6). Then, for each q and $r, \ 1 \leq q \leq r, \ 1 \leq i \leq n$ and all $x_j \in \Omega^N$, the components in the decomposition (6.14) satisfy the following estimates

$$\left|w_{i,q}''(x_j)\right| \leq C \ min\left\{\frac{1}{\sqrt{\varepsilon_q}}, \frac{\sqrt{\varepsilon_q}}{\varepsilon_i}\right\} B_q(x_j),$$

$$\left|w_{i,q}^{(3)}(x_j)\right| \leq C \ min\left\{\frac{1}{\varepsilon_q}, \frac{1}{\varepsilon_i}\right\} B_q(x_j),$$

$$\left|w_{i,r+1}^{(3)}(x_j)\right| \leq C \ min\left\{\sum_{q=r+1}^{n} \frac{B_q(x_j)}{\varepsilon_q}, \sum_{q=r+1}^{n} \frac{B_q(x_j)}{\varepsilon_i}\right\},$$

$$\left|w_{i,q}^{(4)}(x_j)\right| \leq C \frac{B_q(x_j)}{\varepsilon_i\sqrt{\varepsilon_q}}, \ \left|w_{i,r+1}^{(4)}(x_j)\right| \leq C \sum_{q=r=1}^{n} \frac{B_q(x_j)}{\varepsilon_i\sqrt{\varepsilon_q}}.$$

Lemma 6.3 Assume that $d_r > 0$ for some $r, \ 1 \leq r \leq n$. Let $A(x)$ satisfy (5),(6). Then if $x_j \notin J$.

$$\left|\left(L^N(\vec{W} - \vec{w})\right)_i(x_j)\right| \leq C\left[B_r(x_{j-1}) + \ x_{j+1} - x_{j-1}\right] \qquad (6.15)$$

and if $x_j \in J$,

$$\left|\left(L^N(\vec{W} - \vec{w})\right)_i(x_j)\right| \leq CN^{-1} \ln N \ . \qquad (6.16)$$

Lemma 6.4 Let $A(x)$ satisfy (5),(6). Then, on Ω^N, for each $i = 1, \dots, n$,

$$\left|\left(L^N(\vec{W} - \vec{w})\right)_i(x_j)\right| \leq CB_n(x_{j-1}). \qquad (6.17)$$

The following theorem gives the estimate of error in the singular component.

Theorem 6.2 Let $A(x)$ satisfy (5),(6). Let $\vec{w}$ denote the singular component of the solution of the problem (1.1),(1.2) and $\vec{W}$ denote the singular component of the solution of the problem (4.1),(4.2). Then

$$|| \vec{W} - \vec{w} || \leq CN^{-1} \ln N. \qquad (6.18)$$

Proof : From the expression (5.6) and (5.5),

$$\left|\left(\beta_0{}^N(\vec{W} - \vec{w})\right)_i(0)\right| \leq C(x_1 - x_0) max_{s\in[x_0,x_1]}|w_i''(s)| \ \leq \ CN^{-1} \ln N, \qquad (6.19)$$

$$\left|\left(\beta_1{}^N\left(\vec{W} - \vec{w}\right)\right)_i(1)\right| \leq C(x_N - x_{N-1}) max_{s\in[x_{N-1},x_N]}|w_i''(s)| \ \leq \ CN^{-1} \ln N \qquad (6.20)$$

Thus from (6.4),(6.19) and (6.20),

$$\left|\left(\beta_0{}^N\left(\vec{W} - \vec{w}\right)\right)_i(0)\right| \leq \left(\beta_0{}^N\vec{\Phi}\right)_i(0),$$

$$\left|\left(\beta_1{}^N(\vec{W} - w)\right)_i(1)\right| \leq \left(\beta_1{}^N\vec{\Phi}\right)_i(1). \qquad (6.21)$$

In the remaining portion, it is shown that, for all i, j and some constant C,

$$\left|\left(L^N(\vec{W} - \vec{w})\right)_i(x_j)\right| \leq \left(L^N\vec{\Phi}\right)_i(x_j). \qquad (6.22)$$

This is proved for each mesh point $x_j \in \Omega^N$ by considering separately the 8 kinds of subintervals

(a) $(0, \tau_1)$,
(b) $[\tau_1, \tau_2)$,
(c) $[\tau_m, \tau_{m+1})$, for some $m, \ 2 \leq m \leq n-1$.
(d) $[\tau_n, 1/2)$,
(e) $[1/2, 1\text{-}\tau_n]$,
(f) $(1\text{-}\tau_{m+1}, 1 - \tau_m]$, for some $m, \ 2 \leq m \leq n-1$.
(g) $(1 - \tau_2, 1 - \tau_1)]$ and
(h) $(1\text{-}\tau_1, 1)$.

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(a) Clearly, $x_j \notin J$ and $x_{j+1} - x_{j-1} \leq C\sqrt{\varepsilon_1}N^{-1}\ln N$. Then Lemma 6.1 and the expression (6.5) give (6.22). Similar arguments hold for the case (e).

(b) There are 2 possibilities:

(b1) $d_1 = 0$ and

(b2) $d_1 > 0$.

(b1) Since $\tau_1 = \frac{\tau_2}{2}$ and the mesh is uniform in $(0, \tau_2)$, it follows that $x_j \notin J$ and $x_{j+1} - x_{j-1} \leq C\sqrt{\varepsilon_1}N^{-1}\ln N$. Then Lemma 6.1 and expression (6.5) lead to (6.22).

(b2) Either $x_j \notin J$ or $x_j \in J$.

If $x_j \notin J$, then $x_{j+1} - x_{j-1} \leq C\sqrt{\varepsilon_2}N^{-1}\ln N$ and by Lemma 3.1, $B_1(x_{j-1}) \leq B_1^L(x_{j-1}) \leq B_1^L(\tau_1 - h_1^-) \leq CN^{-2}$. Then (6.15) of Lemma 6.3 with $r = 1$ and (6.6) give (6.22).

On the other hand , if $x_j \in J$, then from (6.16) of Lemma 6.3 with $r = 1$, and (6.6) give (6.22) . Similar arguments hold for the case (f).

(c) There are 3 possibilities:

(c1) $d_1 = d_2 = \cdots = d_q = 0$,

(c2) $d_r > 0$ and $d_{r+1} = \cdots = d_q = 0$ for some r, $1 \leq r \leq q-1$ and

(c3) $d_q > 0$.

(c1) Since $\tau_1 = C\tau_{q+1}$ and the mesh is uniform in $(0, \tau_{q+1})$, it follows that $x_j \notin J$ and $x_{j+1} - x_{j-1} \leq C\sqrt{\varepsilon_1}N^{-1}\ln N$. Then Lemma 6.1 and expression (6.5) lead to (6.22).

(c2) Either $x_j \notin J$ or $x_j \in J$.

If $x_j \notin J$, then $\tau_{r+1} = C\tau_{q+1}$, $x_{j+1} - x_{j-1} \leq C\sqrt{\varepsilon_{q+1}}N^{-1}\ln N$ and by Lemma 6.1, $B_r(x_{j-1}) \leq B_r^L(x_{j-1}) \leq B_r^L(\tau_q - h_q^-) \leq B_r^L(\tau_r - h_r^-) \leq CN^{-2}$. Thus, (6.15) of Lemma 6.3 and (6.5) lead to (6.22).

On the other hand , if $x_j \in J$, then $x_j = \tau_q$, so by (6.16) of Lemma 6.3 with $r = q$, and (6.6) lead to (6.22) .

(c3) Either $x_j \notin J$ or $x_j \in J$.

If $x_j \notin J$, then $x_{j+1} - x_{j-1} \leq C\sqrt{\varepsilon_{q+1}}N^{-1}\ln N$ and by Lemma 3.1, $B_q(x_{j-1}) \leq B_q^L(x_{j-1}) \leq B_q^L(\tau_q - h_q^-) \leq CN^{-2}$, so (6.15) of Lemma 6.3 with $r = q$ and (6.5) lead to (6.22).

On the other hand , if $x_j \in J$, then $x_j = \tau_q$. Expression (6.16) of Lemma 6.3 with $r = q$, and (6.6) lead to (6.22) . Similar arguments hold for the case (g).

(d) There are 3 possibilities:

(d1) $d_1 = d_2 = \cdots = d_n = 0$,

(d2) $d_r > 0$ and $d_{r+1} = \cdots = d_n = 0$ for some r, $1 \leq r \leq n-1$ and

(d3) $d_n > 0$.

(d1) Since the mesh is uniform in $[0,1]$, it follows that $x_j \notin J$, $\frac{1}{\sqrt{\varepsilon_1}} \leq C\ ln\ N$ and $x_{j+1} - x_{j-1} \leq CN^{-1}$. Then Lemma 6.1 and expression (6.5) lead to (6.22).

(d2) Either $x_j \notin J$ or $x_j \in J$.

If $x_j \notin J$, then $\frac{1}{\sqrt{\varepsilon_{r+1}}} \leq C\ln N$, $x_{j+1} - x_{j-1} \leq CN^{-1}$ and by Lemma 3.1, $B_r(x_{j-1}) \leq B_r^L(x_{j-1}) \leq B_r^L(\tau_n - h_n^-) \leq B_r^L(\tau_r - h_r^-) \leq CN^{-2}$. Then (6.15) of Lemma 6.3 and (6.5) give (6.22).

On the other hand , if $x_j \in J$, then $x_j \in \{\tau_n, 1-\tau_n, \ldots, 1-\tau_1\}$. Then expression (6.16) of Lemma 6.3 and (6.3) lead to (6.22) .

(d3) By Lemma 6, with $r = n$, $B_n(x_{j-1}) \leq B_n^L(x_{j-1}) \leq B_n(\tau_n - h_n^-) \leq CN^{-2}$. Then Lemma 6.4 and (6.5) give (6.22). Similar arguments hold for the case (h).

By using Lemma 4.3, the result is established from (6.21) and (6.22).

The following theorem gives the required essentially first order parameter-uniform error estimate.

Theorem 6.3 Let $A(x)$ satisfy (1.5),(1.6). Let $\vec{u}$ denote the solution of the problem (1.1),(1.2) and $\vec{U}$ be the solution of the problem (4.1),(4.2). Then,

$$\|\vec{U} - \vec{u}\| \leq CN^{-1}\ln N. \qquad (6.23)$$

Proof: An application of the triangular inequality and the results of Theorems 6.1 and 6.2 lead to the required result.

VII. NUMERICAL ILLUSTRATION

The numerical method proposed above is illustrated through an example presented in this section. The method proposed above is applied to solve the problem and parameter-uniform order of convergence and the parameter-uniform error constants are computed. The numerical results for $\vec{u}$ are presented in Table 1.

Example 7.1: Consider the boundary value problem

$$-E\vec{u}''(x) + A(x)\vec{u}(x) = \vec{f}(x), \text{ for } x \in (0,1),$$

with

$$\vec{u}(0) - \vec{u}'(0) = \vec{\phi}_0, \quad \vec{u}(1) + \vec{u}'(1) = \vec{\phi}_1$$

where $E = diag\,(\varepsilon_1, \varepsilon_2, \varepsilon_3)$,

$$A(x) = \begin{pmatrix} 4 & -2+x & -1 \\ -2x & 5+x & -1 \\ -1 & -2 & 4+x \end{pmatrix}, \vec{f} = (1+x, e^x, 1)^T,$$

$\vec{\phi}_0 = (1,1,1)^T$, $\vec{\phi}_1 = (1,1,1)^T$.

As in [4], the notations D^N, p^N and C_p^N denote the $\vec{\varepsilon}$ −uniform maximum pointwise two-mesh differences, the $\vec{\varepsilon}$ −uniform order of convergence and the $\vec{\varepsilon}$ −uniform error constant respectively and are given by $D^N = max_{\vec{\varepsilon}}\, D_{\vec{\varepsilon}}^N$ where $D_{\vec{\varepsilon}}^N = \|\vec{U}_{\vec{\varepsilon}}^N - \bar{U}_{\vec{\varepsilon}}^{2N}\|_{\Omega^N}$, $p^N = \log_2 \frac{D^N}{D^{2N}}$ and $C_p^N = \frac{D^N N^{p^*}}{1-2^{-p^*}}$. Then the parameter-uniform order of convergence and the error constant are given by $p^* = \min_N p^N$ and $C_{p^*}^N = \max_N C_p^N$ respectively. It is evident from the Figure 1 & 2 that the solution $\vec{u}$ exhibits no layers whereas the derivative $\vec{u}'$ exhibits boundary layers at 0 & 1. Further, The order of convergence of $\vec{u}$ presented in Table 1 agree with the theoretical results.

Figure 1

The numerical approximation of $\vec{u}$ for $\varepsilon_1 = 2^{-16}$, $\varepsilon_2 = 2^{-15}$, $\varepsilon_3 = 2^{-14}$ and N = 384

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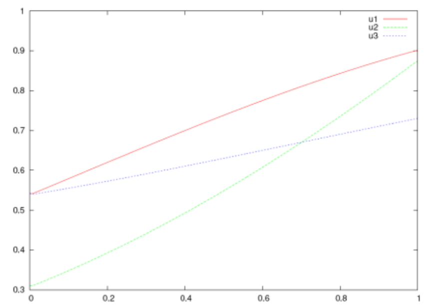

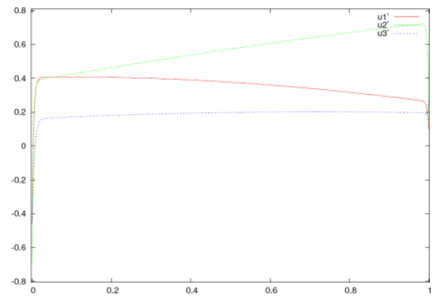

Figure 2

The numerical approximation of $\vec{u}'$ for $\varepsilon_1 = 2^{-16}$, $\varepsilon_2 = 2^{-15}$, $\varepsilon_3 = 2^{-14}$ and N = 384

Table 1

Values of $D_\varepsilon^N, D^N, p^N, p^*$ and $C_{p^*}^N$ for $\varepsilon_1 = \frac{\eta}{128}, \varepsilon_2 = \frac{\eta}{64}, \varepsilon_3 = \frac{\eta}{32}, \alpha = 0.9$

η	Number of mesh points N				
	48	96	192	384	768
2^{-3}	0.373E-02	0.173E-02	0.833E-02	0.408E-02	0.202E-03
2^{-6}	0.269E-02	0.147E-02	0.896E-03	0.441E-02	0.214E-03
2^{-9}	0.971E-03	0.528E-03	0.286E-03	0.157E-03	0.859E-04
2^{-12}	0.346E-03	0.188E-03	0.102E-03	0.556E-04	0.305E-04
2^{-15}	0.123E-03	0.665E-04	0.361E-04	0.197E-04	0.108E-04
D^N	0.373E-02	0.173E-02	0.896E-03	0.441E-03	0.214E-03
p^N	0.111E+01	0.952E+00	0.102E+01	0.104E+01	
C_p^N	0.308E+00	0.277E+00	0.277E+00	0.264E+00	0.248E+00
Computed order of $\vec{\varepsilon}$- uniform convergence, p^*= 0.9523636E + 00					
Computed order of $\vec{\varepsilon}$- uniform error constant, $C_{p^*}^N$= 0.3084886E + 00					

VIII. CONCLUSION

A singularly perturbed linear system of second order ordinary differential equations of reaction-diffusion type with Robin boundary conditions is considered. From the figure 1, it is evident that the components of the solution $\vec{u}$ exhibit no layers and from figure 2, the components of the solution $\vec{u}'$ exhibit twin layers at 0 and 1. The numerical approximations obtained with finite difference scheme on Shishkin mesh are proved essentially first order convergent. The order of convergence of $\vec{u}$ presented in Table 1 agree with the theoretical results.

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