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Intuitionistic Fuzzy Magdm Problems With Numerical Solution Of Singularly Perturbed Differential Equation Of Convection-Diffusion Type

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Abstract **In this paper, the Multiple Attribute Group Decision Making (MAGDM) problems with intuitionistic fuzzy sets are considered. The weights of the decision maker are provided in the form of a singularly perturbed differential equation of convection diffusion type. The weights are calculated using the exact and numerical methods by the construction of Shishkin mesh and a finite difference scheme and it is applied in MAGDM problems under intuitionistic fuzzy environment. A new correlation coefficient for intuitionistic fuzzy sets is proposed which is used for ranking the alternatives. A class of operators based on Intuitionistic Fuzzy Ordered Weighted Geometric (IFOWG) operator is utilized for aggregating the attributes in contrast with the alternatives together with the decision maker weights to identify the best alternative from the available ones. Numerical illustration is given to show the effectiveness of the proposed approach.**

Keywords- MAGDM, Intuitionistic Fuzzy Sets, Intuitionistic Fuzzy Ordered Weighted Geometric (IFOWG) operator, Intuitionistic Fuzzy Hybrid Geometric (IFHG) operator, singularly perturbed differential equations, Shishkin Mesh

I. INTRODUCTION

Since the theory of Fuzzy Sets (FSs) was proposed by **Zadeh [30],** many research achievements based on FSs have been obtained. However, because FSs are based on membership functions, they cannot express non-membership degrees. **Atanassov [1-3]** proposed the Intuitionistic Fuzzy Set (IFS) which are characterized by a membership function and a non-membership function, which is a generalization of the concept of FS. **Chen & Yang [4]** investigated the MAGDM problem with intuitionistic fuzzy information which is very useful for solving complicated decision problems using score values under uncertain circumstance. **Liu & Chen [7]** provide the general operational rule for intuitionistic fuzzy number. **Robinson & Amirtharaj [12-17], Jeeva & Robinson [6]** and **Robinson & Jeeva [18]** discussed the various decision making operators and proposed correlation coefficients for different higher order intuitionistic fuzzy sets and utilized them in ranking the alternatives in MAGDM problems. **Robinson & Indhumathi [19]** determined the unknown weights using Singularly Perturbed Delay Differential Equations and proposed a new correlation coefficient for IFSs and utilized them in ranking the alternatives in MAGDM problems. **Zeng & Li [29]** and **Gerstenkorn & Manko [5]** discussed the correlation coefficient of intuitionistic fuzzy set. **Szmidt & Kacprzyk [21, 22]** proposed some solution like the intuitionistic fuzzy core and consensus winner in group decision making with intuitionistic fuzzy preference relations. They also developed an approach to aggregate the individual intuitionistic fuzzy preference relations into a social fuzzy preference relation based on fuzzy majority equated with a fuzzy linguistic quantifier. **Xu [23], Xu & Da [26], Xu & Yager [27], Xu & Chen [24, 25]** and **Yager [28]** developed the OWA, OWG and Induced OWA(IOWA) operators for MAGDM problems with intuitionistic fuzzy information. Using OWA operator, one can order the weight either in ascending or in a descending order depending upon the data values but for IOWA one can use the same procedure of OWA where the only difference is that the weight can be ordered through the order inducing variables.

In this work, Singular Perturbation Problem (SPP) are used for determining weights of decision makers in MAGDM problems. First to distinguish between the regular perturbation problems and the singular perturbation problems, consider a family of boundary value problems (BVPs) P_ε, depending on a small parameter ε. Under certain conditions, the solution

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$y_\varepsilon(x)$ of P_ε can be constructed by a well-known 'method of perturbation'; that is, as a power series in ε with its first term y_0 being the solution of the problem P_0 (obtained by putting ε=0 in P_ε). When such a power series expansion converges as $\varepsilon \to 0$ uniformly in x then it is a regular perturbation problem. When $y_\varepsilon(x)$ does not have a uniform limit in x as $\varepsilon \to 0$ this regular perturbation method fails and is called a singular perturbation problem [8-11]. Typically these problems arise in various fields of applied mathematics such as fluid dynamics (boundary layer problems), elasticity (edge effect in shells), quantum mechanics (WKB problems), electrical networks, chemical reactions, control theory, gas porous electrodes theory and many other areas. The Navier-Stokes equation with a large Reynolds number is one of the most striking examples of SPP, which led to the idea of boundary layer, introduced by L. Prandtl. Convective heat transport problem with large peclet number is another important example to be noted. Due to the presence of the perturbation parameter ε, the solutions of singularly perturbation equations and /or their derivatives behave non-smooth in some portion of the domain of definition of the problems. In those sub-domains the solutions or their derivatives exhibit boundary or interior layers.

The concept of boundary layer was introduced by Prandtl at the Third International congress of mathematicians in Heidelberg in 1904 and it was reported in the proceedings of the conference. In his paper fluid motion with very small friction, read before the mathematical congress, Prandtl proved that the flow about a body can be treated by dividing the domain of flow into two regions: a very thin layer near the body which he called as boundary layer, where frictional effects are prominent and the remaining as the outer region. On the basis of this hypothesis, Prandtl emphasized the importance of viscous flows without delving into the mathematical complexities involved. This boundary layer theory became the foundation stone for modern fluid dynamics. Numerical analysis and asymptotic analysis are two principal approaches for solving singular perturbation problems [8-11]. Since the goals and the problem classes are rather different, there has not been much interaction between these approaches. Numerical analysis tries to provide quantitative information about a particular problem. Whereas asymptotic analysis tries to provide an insight into the qualitative behaviour of a family of problems and semi quantitative information about any particular member of the family.

The numerical treatment of SPPs has attracted a good number of scientists for the past six decades. It is well known that the solutions of SPPs are non-smooth with singularities related to boundary layers. When the perturbation parameter ε is close to a critical value, even the most contemporary numerical methods fail to be robust and layer-resolving. Careful examination of the numerical results from the various finite difference schemes on uniform grids show that, for fixed (small) value of the parameter ε, the maximum point wise error usually increases as the mesh is refined until the mesh parameter and the perturbation parameter ε have the same order of magnitude. This is due to the presence of the so called 'boundary or interior layers exhibited by the solution. Of all the numerical methods suggested for SPPs the most popular methods based on finite differences are fitted operator methods and fitted mesh methods. A fitted method uses simply a classical finite difference operator on a piecewise-uniform mesh fitted on the domain of definition of the differential equation. Hence one has to look for robust computational methods which will give numerical approximations which inherit the stability properties of the exact solution by preserving the monotonicity of the original problem.

Malley [8] and **Nayfeh [11]** gave an introduction to singular perturbation problems. **Ross et al. [20]** and **Matthews et al. [9]** presented a general introduction to parameter-uniform numerical methods for singular perturbation problems. **Miller et al. [10]** in their work have devoted the last five chapters for SPPs in two dimension. Fitted mesh methods and their parameter uniform convergence have been established for these problems. They have also proved that it is impossible to construct parameter-uniform numerical method using a standard finite difference operator on a uniform rectangular mesh for a problem having both an initial and parabolic boundary layer. They have suggested both a finite difference operator and a piecewise-uniform fitted mesh to achieve the parameter-uniform numerical method. In this paper, we have investigated the MAGDM problem with intuitionistic fuzzy set for ranking the alternatives together with IFWG and IFHG operators. A new correlation coefficient of IFSs is proposed for ranking the best alternatives. The decision maker has provided the weight vector information in the form of a singularly perturbed differential equation of convection-diffusion type. This differential equation is solved through numerical methods by constructing the Shishkin mesh and finite difference methods. The numerical solution to the differential equation is normalized and hence the weights of the decision maker are derived. The derived decision maker weights are applied in the decision problem for further aggregation of the IFS information which is given in the form of a decision matrix. A numerical illustration is given to show the effectiveness of the proposed approach.

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II. Basic Concepts of Intuitionistic Fuzzy Set

Let X be the universe of discourse. An intuitionistic fuzzy set A in X is an object having the form $A=\{\langle x,\mu_A(x),\gamma_A(x)\rangle | \forall x\in X\}$ where $\mu_A(x),\gamma_A(x): x\to[0,1]$ denote membership function and non-membership function, respectively, of A and satisfy $0\le\mu_A(x)+\gamma_A(x)\le 1$ for every $x\in X$. $\mu_A(x)$ is the lowest bound of membership degree derived from proofs of supporting *x*; $\gamma_A(x)$ is the lowest bound of non-membership degree derived from proofs of rejecting *x*. It is clear that the membership degree of Intuitionistic Fuzzy set *A* has been restricted in $[\mu_A(x),1-\gamma_A(x)]$ which is a subinterval of [0, 1]. For each IFS *A* in *X* we call $\pi_A(x)=1-\mu_A(x)-\gamma_A(x)$ as the intuitionistic index of *x* in *A*. It is hesitation degree (or degree of indeterminacy) of *x* to *A*. It is obvious that $0\le\mu_A(x)\le 1$ for each $x\in X$.

For example, let *A* be a IFS with membership function $\mu_A(x)$ and non-membership function $\gamma_A(x)$, respectively. If $\mu_A(x)=0.5$ and $\gamma_A(x)=0.3$, then we have $\mu_A(x)=1-0.5-0.3=0.2$. It could be interpreted as the degree that the object *x* belongs to the IFS *A* is 0.5, the degree that the object *x* does not belong to the IFS *A* is 0.3 and the degree of hesitation is 0.2. Thus, the IFS *A* in *X* can also be expressed as $A=\{\langle x,\mu_A(x),\gamma_A(x),\pi_A(x)\rangle : x\in X\}$

If *A* is an ordinary fuzzy set, then $\pi_A(x)=1-\mu_A(x)-(1-\mu_A(x))=0$ for each $x\in X$. It means that the third parameter $\pi_A(x)$ cannot be casually omitted if *A* is a general IFS, not an ordinary fuzzy set. Therefore, the representation of IFS should consider all three parameters in calculating the degree of similarity between IFSs. For $A,B\in IFS(X)$, the set of all IFSs, the notion of containment is defined as follows: $A\subseteq B \Leftrightarrow \mu_A(x)\le\mu_B(x)$ and $\gamma_A(x)\ge\gamma_B(x), x\in X$

Definition: (Intuitionistic Fuzzy Set).

An IFS A in X is given by $A=\{\langle x,\ \mu_A(x),\ \gamma_A(x)\rangle / x\in X\}$, where $\mu_A: X\to[0,1]$, $\gamma_A: X\to[0,1]$, with the condition $0\le\mu_A(x)+\gamma_A(x)\le 1,\ \ \forall x\in X$ .The numbers $\mu_A(x)$ and $\gamma_A(x)$ represent, the membership degree and non-membership degree of the element x to the set A, respectively.

Definition: For each IFS A in X, if $\pi_A(x)=1-\mu_A(x)-\gamma_A(x)$, $\forall x\in X$, then $\pi_A(x)$ is called the degree of indeterminacy or hesitancy of *x* to *A*, where $0\le\pi_A(x)\le 1$, for all $x\in X$.

III. Different Classes of Aggregation Operators in Intuitionistic Fuzzy Set

Geometric Mean (*GM*) Operator:

Definition: A Geometric Mean (*GM*) operator of dimension m is a mapping $GM: R^m\to R$ and is defined as:

$$GM(a_1,a_2,...,a_m)=\prod_{j=1}^{m}\left(a_j\right)^{1/m}$$

Ordered Weighted Geometric (*OWG*) Operators:

The Ordered Weighted Geometric (*OWG*) operator is based on the *OWA* operator and the Geometric Mean (*GM*) operator and provides a parameterized family of aggregation operators used in many applications. The definition of the *OWG* operator is as follows:

Definition: An *OWG* operator of dimension *m* is a mapping $OWG: R^m\to R$ that has an associated weighting vector $w=(w_1,w_2,...,w_m)^T$ of dimension *m* having the properties, $w_j\in[0,1]$, $\sum_{j=1}^{m} w_j=1$ and such that

$$OWG(a_1,a_2,...,a_m)=\prod_{j=1}^{m} b_j^{w_j}$$

Where b_j is the j^{th} largest of the a_i .

Example: Assume $w=(0.4,0.3,0.2,0.1)^T$
Consider the decision data given by $a_j=(0.7,0.1,0.2,0.6)$. Then

$$OWG\ (a_j)=(0.1^{0.4})\times(0.7^{0.3})\times(0.6^{0.2})\times(0.2^{0.1}),$$
$$=0.69$$

A fundamental aspect of this operator is the reordering of the arguments, based upon their values. That is, the weights rather than being associated with a specific argument, as in the case of the usual weighted average, are associated with a particular position in the ordering.

Note:

Different *OWG* operators are distinguished by their weighting functions. The three important special cases of *OWG* aggregations are:

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- Maximum: in this case $w^* = (1,0,0,...,0)^T$

$$OWG(a_1, a_2, ..., a_m) = \max_i(a_i)$$

- Minimum: in this case $w_* = (0,0,0,...,1)^T$

$$OWG(a_1, a_2, ..., a_m) = \min_i(a_i)$$

- Average: in this case $w_A = \left(\frac{1}{m}, \frac{1}{m}, ..., \frac{1}{m}\right)^T$

$$OWG(a_1, a_2, ..., a_m) = \prod_i (a_i)^{\frac{1}{m}}$$

- The *OWG* operator is: Commutative, Monotonic and Idempotent.

Definition: Intuitionistic Fuzzy Weighted Geometric Operator (IFWG)

Let $a_j = (\mu_j, \gamma_j)$, for all $j = 1,2,........,n$ be a collection of intuitionistic fuzzy values. The Intuitionistic Fuzzy Weighted Geometric (IFWG) operator, $IFWG : Q^n \to Q$ is defined as:

$$IFWG_\omega(a_1, a_2, ..., a_n) = \sum_{j=1}^{n} \omega_j a_j$$

$$= (\prod_{j=1}^{n} \mu_j^{\omega_j}, 1 - \prod_{j=1}^{n} (1-\gamma_j)^{\omega_j}),$$

where $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ be the weight vector of a_j, for all $j = 1,2,...,n$ such that $\omega_j > 0$ and $\sum_{j=1}^{n} \omega_j = 1$.

Definition: Intuitionistic Fuzzy Ordered Weighted Geometric Operator (IFOWG)

Let $\tilde{a}_j = (\mu_j, \gamma_j)$, for all $j = 1,2,...,n$ be a collection of intuitionistic fuzzy values. The Intuitionistic Fuzzy Weighted Geometric (IFOWG) operator, $IFOWG : Q^n \to Q$ is defined as:

$$IFOWG_\omega(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \sum_{j=1}^{n} \tilde{a}_{\sigma(j)}^{\omega_j}$$

$$= \left(\prod_{j=1}^{n} \mu_{\sigma(j)}^{\omega_j}, 1 - \prod_{j=1}^{n} (1-\gamma_{\sigma(j)})^{\omega_j}\right),$$

where $w = (w_1, w_2, ..., w_n)^T$ is the associated weight vector such that $w_i > 0$ and $\sum_{j=1}^{n} w_j = 1$. Furthermore, $(\sigma(1), \sigma(2), ..., \sigma(n))$ is a permutation of (1,2,…, n), such that $\tilde{a}_{\sigma(j-1)} \geq \tilde{a}_{\sigma(j)}$ for all j=2,…, n.

Definition: Intuitionistic Fuzzy Hybrid Geometric Operator (IFHG)

Let $\tilde{a}_j = (\mu_j, \gamma_j)$, for all $j = 1,2,...,n$ be a collection of intuitionistic fuzzy values. The Intuitionistic Fuzzy Hybrid Geometric (IFHG) operator, $IFHG : Q^n \to Q$ is defined as:

$$IFHG_{\omega,w}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \sum_{j=1}^{n} \tilde{a}_{\sigma(j)}^{w_j}$$

$$= \left[\prod_{j=1}^{n} (\mu_{a_{\sigma(j)}})^{w_j}, 1 - \prod_{j=1}^{n} (1-\gamma_{a_{\sigma(j)}})^{w_j}\right],$$

where $w = (w_1, w_2, ..., w_n)^T$ is the associated vector such that $w_j > 0$ and $\sum_{j=1}^{n} w_j = 1$, and where $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ be the weight vector of a_j, for all $j = 1,2,...,n$ such that $\omega_j > 0$ and $\sum_{j=1}^{n} \omega_j = 1$.

Furthermore $a_{\sigma(j)}$ is the j^{th} largest of the weighted intuitionistic fuzzy numbers $a_j = a_j^{n\omega_j}$, $j = 1,2,...,n.$

IV. Correlation Coefficient of Intuitionistic Fuzzy Sets (IFSs)

By correlation analysis, the joint relationship of two variables can be examined with a measure of interdependence of the two variables. It is well known that the conventional correlation analysis using probabilities and statistics was inadequate to handle uncertainty of failure data and modeling. The method to measure the correlation between two variables involving fuzziness is a challenge to classical statistical theory. Fuzzy correlation has captured the attention of researchers recently. Correlation coefficient of fuzzy sets, interval-valued fuzzy sets, intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets are already in literature. Various attempts were made by researchers in the recent days to define the correlation coefficient of intuitionistic fuzzy sets and Interval-valued Intuitionistic Fuzzy sets. As vague sets deal with truth membership value, false membership value and the vague degree, and have more ability to deal with uncertain information than traditional fuzzy sets [12, 15], many researchers pay attention on vague set theory. **Zeng & Li [29]** focused on probability spaces to define a new kind of correlation for intuitionistic fuzzy sets. **Gerstenkon & Manko [5]** defined the correlation of intuitionistic fuzzy sets as an ensemble of ordinary fuzzy set, and defined correlation coefficient of intuitionistic fuzzy sets by using the correlation coefficient of two ordinary fuzzy sets and a mean aggregation function. In their definition, the correlation coefficient lies between 0

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and 1, differing from the conventional range of [–1, 1]. In this paper, a new method to calculate the correlation coefficient of intuitionistic fuzzy sets is proposed based on the method proposed by **Robinson & Amirtharaj [12]** for calculating correlation coefficient of vague sets, taking the membership, non-membership and the hesitancy grades into account.

Let $X = \{x_1, x_2, ..., x_n\}$ be the finite universal set and $A, B \in$ IFS(X) be given by

$$A = \{\langle x, [\mu_A(x), \gamma_A(x)]\rangle \ / x \in X\},$$

$$B = \{\langle x, [\mu_B(x), \gamma_B(x)]\rangle \ / x \in X\}.$$

And the length of the intuitionistic fuzzy value values are given by:

$$\pi_A(x) = 1 - \mu_A(x) - \gamma_A(x),$$
$$\pi_B(x) = 1 - \mu_B(x) - \gamma_B(x).$$

Now for each $A \in$ IFS(X), the informational intuitionistic energy of A is defined as follows:

$$E_{IFS}(A) = \frac{1}{n}\sum_{i=1}^{n}\left[\mu_A^2(x_i) + (1-\gamma_A(x_i))^2 + \pi_A^2(x)\right],$$ and for each $B \in$ IFS(X), the informational intuitionistic energy of B is defined as follows:

$$E_{IFS}(B) = \frac{1}{n}\sum_{i=1}^{n}\left[\mu_B^2(x_i) + (1-\gamma_B(x_i))^2 + \pi_B^2(x)\right],$$

The correlation between the IFSs A and B is given by the formula:

$$C_{IFS}(A,B) = \frac{1}{n}\sum_{i=1}^{n}\begin{bmatrix}\mu_A(x_i)\mu_B(x_i) + (1-\gamma_A(x_i)) \\ (1-\gamma_B(x_i)) + \pi_A(x)\pi_B(x)\end{bmatrix},$$

Furthermore, the correlation coefficient between the IFSs A and B is defined by the formula:

$$K_{IFS}(A, B) = \frac{C_{IFS}(A, B)}{\sqrt{E_{IFS}(A) \cdot E_{IFS}(B)}},$$

where $0 \le K_{IFS}(A,B) \le 1$.

Proposition 1:

For $A, B \in$ IFS(X), we have:

i) $0 \le C_{IFS}(A,B) \le 1,$

ii) $C_{IFS}(A,B) = C_{IFS}(B,A),$

iii) $K_{IFS}(A,B) = K_{IFS}(B,A).$

Theorem 1: For A, $B \in$ IFS(X), then $0 \le K_{IFS}(A, B) \le 1.$

Theorem 2: $K_{IFS}(A,B) = 1 \Leftrightarrow A = B.$

Theorem 3: $C_{IFS}(A,B) = 0 \Leftrightarrow A$ and B are non-fuzzy sets and satisfy the condition $\mu_A(x_i) + \mu_B(x_i) = 1$ or $\gamma_A(x_i) + \gamma_B(x_i) = 1$ or $\pi_A(x_i) + \pi_B(x_i) = 1, \ \forall x_i \in X.$

Theorem 4: $C_{IFS}(A,A) = 1 \Leftrightarrow A$ is a non-fuzzy set.

V. SINGULAR PERTURBATION PROBLEMS

In a differential equation, a small positive parameter ε multiplying the highest order derivative and/or the lower order derivatives is known as the singular perturbation problems. A Singular perturbation problem is said to be convection-diffusion type if the order of the differential equation is reduced by one when the perturbation parameter ε is set to equal to zero.

We consider a class of linear singular perturbation problems of the form

$-\varepsilon u_\varepsilon''(x) + b(x)u_\varepsilon'(x) = f(x), \quad 0 < x < 1$ with boundary condition $u_\varepsilon(0) = u_0, u_\varepsilon(1) = u_1$.

Where u_0, u_1 are given constants, ε is a small parameter $0 < \varepsilon \le 1$; b(x) and f(x) are continuous on [0, 1]. It is assumed furthermore that the coefficient function satisfies the condition $a(x) > \alpha > 0$ for all $x \in \bar{\Omega}$.

The differential operator L_ε for the above problem is defined, for all $\psi \in c^2(\bar{\Omega})$, by $L_\varepsilon \psi \equiv -\varepsilon\psi'' + a\psi'$.

To support the numerical experiments we are making use of the results in [10].

3.1 Analytical Results

The operator L_ε satisfies the following maximum principle:

Lemma 1:

Assume that $\psi(0) \ge 0$ *and* $\psi(1) \ge 0$. Then, $L_\varepsilon \psi(x) \ge 0$ for all $x \in \bar{\Omega}$ implies that $\psi(x) \ge 0$ for all $x \in \bar{\Omega}$.

As a consequence of the maximum principle, there is established the stability result for the above problem in the following:

Lemma 2:

If u_ε is any function in C, then for all $x \in [0,1]$, $|u_\varepsilon(x)| \le C(1+x)$.

Lemma 3:

Let u_ε be the solution of the above problem. Then, for $0 \le k \le 3$,

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$$\left|u_\varepsilon^{(k)}(x)\right| \le C(1+\varepsilon^{-k}e^{-\alpha(1-x)/\varepsilon})$$

for all $x \in \bar{\Omega}$.

The Shishkin decomposition of the solution u_ε has to be decomposed into smooth and singular components as follows:

$u_\varepsilon = v_0 + \varepsilon y_1 + w_0$.

Where the smooth component v_0 is the solution of

$L_\varepsilon y_1 = v_0''$, $y_1(0) = -\varepsilon^{-1} w_0(0)$, $y_1(1) = 0$, and the singular component w_0 is the solution of

$L_\varepsilon w_0 = 0$, $w_0(0) = w_0(1)e^{-\alpha/\varepsilon}$, $w_0(1) = u_1 - v_0(1)$.

Theorem 5:

The solution u_ε of the above problem has the decomposition $u_\varepsilon = v_\varepsilon + w_\varepsilon$ where, for all k, $0 \le k \le 3$, and all $x \in \bar{\Omega}$, the smooth component v_ε satisfies

$$\left|v_\varepsilon^{(k)}(x)\right| \le C(1+\varepsilon^{-(k-2)}e^{-\alpha(1-x)/\varepsilon}),$$

and the singular component w_ε satisfies

$$\left|w_\varepsilon^{(k)}(x)\right| \le C\varepsilon^{-k}e^{-\alpha(1-x)/\varepsilon},$$

for some constant C independent of ε.

5.2 Shishkin Mesh

On the domain of definition, a piecewise uniform mesh is to be constructed. As the solutions exhibit boundary layer at x=1, the mesh is to be fine in the neighborhood f x=1 and coarse elsewhere. If the total number of mesh points is N, then N/2 points are distributed among the inner domain and the remaining N/2 in the outer region. The mesh is precisely as presented below:

$$x_j = x_0 + jh_1 \quad for \ 1 \le j \le \frac{N}{2}.$$

$$x_{\frac{N}{2}+j} = (1-\tau) + jh_2 \quad for \ 1 \le j \le \frac{N}{2}.$$

Where the parameter τ is defined as:

$$\tau = \min\left\{\frac{1}{2}, \frac{\varepsilon}{\alpha}\ln N\right\} \text{ and } h_1 = \frac{2(1-\tau)}{N}, \ h_2 = \frac{2\tau}{N}.$$

5.3 Finite difference Methods

The classical finite difference operator with an appropriate piecewise uniform mesh is used to discrete the above boundary value problem is presented as follows:

$$L_\varepsilon^N = -\varepsilon\delta^2 U(x_i) + b(x_i)D^+U(x_i) = f(x_i), \quad \forall x \in (0,1).$$

Where

$$\delta^2 U(x_i) = \frac{(D^+ - D^-)}{\bar{h}}U(x_i), \quad \bar{h} = \frac{h_i + h_{i+1}}{2},$$

$$D^+U(x_i) = \frac{U(x_{i+1}) - U(x_i)}{h_{i+1}};$$

$$D^-U(x_i) = \frac{U(x_i) - U(x_{i-1})}{h_i}.$$

This is used to compute numerical approximation to the solution of the above problem. The following discrete results are analogous to those for the continuous case.

Lemma 4: Assume that the mesh function Ψ_i satisfies $\Psi_0 \ge 0 \ and \ \Psi_N \ge 0.$

Then $L_\varepsilon^N \Psi_i \ge 0 \ for \ 1 \le i \le N-1$ implies that $\Psi_i \ge 0$ for all $0 \le i \le N$.

Lemma 5:

If Z_i is any mesh function such that $Z_0 = Z_N = 0$, then

$$|Z_i| \le \frac{1}{\alpha}\max_{1\le j\le N-1}\left|L_\varepsilon^N Z_j\right| \quad for \quad 0 \le i \le N.$$

Theorem 6:

The solution u_ε of the continuous problem and the solution U_ε of the discrete problem satisfy the following ε-uniform error estimate

$$\sup_{0<\varepsilon\le 1}\left\|U_\varepsilon - u_\varepsilon\right\| \le CN^{-1}(lN)^2.$$

VI. Weight Vector Determination using Singular Perturbation Problem

Problem proposed by the decision maker:

The following problem is proposed by the decision maker instead of giving direct weights to the decision variables. The problem is solved through coding in FORTRAN in the LINUX environment:

$\varepsilon u''(x) + u'(x) = 0$ with $u(0) = 0, u(1) = 1;$ $0 < x < 1.$

Solution:

The exact solution of the above problem is:

$$y(x) = \left[\frac{1 - e^{\left(\frac{-x}{\varepsilon}\right)}}{1 - e^{\left(\frac{-1}{\varepsilon}\right)}}\right].$$

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And the numerical solution is calculated by using the above finite difference scheme and fix ε =0.01.

Table: 1 Exact solution for $-\varepsilon u''(x)+u'(x)=0$

N	Average of exact solution	Normalization of exact solution
256	0.99338	0.33305
512	0.99453	0.33343
1024	0.99476	0.33351

Figure: 1 Exact solution for $-\varepsilon u''(x)+u'(x)=0$

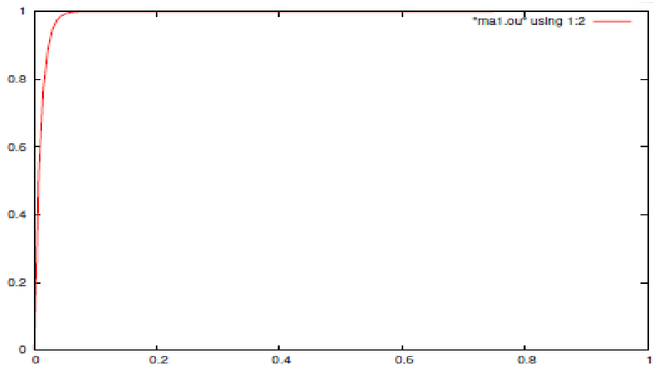
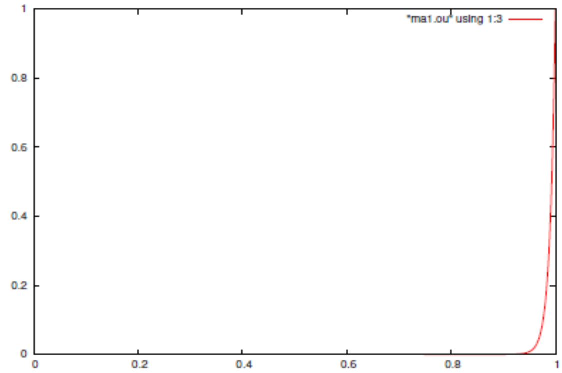


Table: 2 Numerical solution for $-\varepsilon u''(x)+u'(x)=0$

N	Average of numerical solution	Normalization of numerical Solution
256	0.05127	0.20452
512	0.07408	0.29550
1024	0.06588	0.26279
2048	0.05949	0.23731

Figure: 2 Numerical solution for $-\varepsilon u''(x)+u'(x)=0$



VII. Algorithm for Group Decision Making with Intuitionistic Fuzzy Information

Let $A=\{A_1,A_2,...,A_n\}$ be a discrete set of alternatives, and $G=\{G_1,G_2,...,G_n\}$ be the set of attributes, $\omega=\{\omega_1,\omega_2,...,\omega_n\}$ is the weighting vector of the attribute G_j for all j=1,2,...,n, where $\omega_j\in[0,1]$, $\sum_{j=1}^{n}\omega_j=1$.Let $D=\{D_1,D_2,...,D_n\}$ be the set of decision makers, $w=(w_1,w_2,...,w_n)$ be the weighting vector of decision makers, with $w_k\in[0,1]$, $\sum_{k=1}^{n}w_k=1$, and $v=(v_1,v_2,...,v_n)$ be the weighting vector of order inducing variable, with $v_i\in[0,1]$, $\sum_{i=1}^{n}v_i=1$. Suppose that $\tilde{R}_k=\left(\tilde{r}_{ij}^{(k)}\right)_{m\times n}=\left(\mu_{ij}^{(k)},\gamma_{ij}^{(k)}\right)_{m\times n}$ is the intuitionistic fuzzy decision matrix, where $\left(\mu_{ij}^{(k)}\right)$ indicates the degree that the alternative A_i satisfies the attribute G_j given by the decision maker D_k, $\left(\gamma_{ij}^{(k)}\right)$ indicates the degree that the alternative A_i does not satisfy the attribute G_j given by the decision maker D_k, $\mu_{ij}^{(k)}\subset[0,1],\gamma_{ij}^{(k)}\subset[0,1]$ and $\mu_{ij}^{(k)}+\gamma_{ij}^{(k)}\le 1$, for i=1,2,..., m, j=1,2,..., n, k=1,2,…,t.

Then the algorithm for MAGDM using IFOWG and IFHG operator is as follows:

Step 1. Utilize the IFOWG operator to aggregate all individual intuitionistic fuzzy decision matrices $R^{(k)}=\left(r_{ij}^{(k)}\right)_{m\times n}$ $(k=1,2,3,4)$ into a collective intuitionistic fuzzy decision matrix $R=(r_{ij})_{m\times n}$.

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Step 2. Utilize the IFHG operator, $\tilde{r}_i = (\mu_i, \gamma_i) = IFHG_{v,w} = \left(\tilde{r}_i^{(1)}, \tilde{r}_i^{(2)}, \ldots, \tilde{r}_i^{(t)}\right),$

$$i = 1, 2, \ldots m$$

to derive the collective overall preference intuitionistic fuzzy values $\tilde{r}_i (i = 1, 2, \ldots .. m)$ of the alternative A_i along with the weighting vectors of the decision maker derived from solving the singularly perturbed differential equation of convection diffusion type.

Step 3. Calculate the correlation coefficient between the collective overall preference values r_i and the positive ideal value $\tilde{r}^+$, where $\tilde{r}_i = (1, 0)$.

The correlation coefficient between the IFSs, $A, B \in IFS(x)$ is given by the formula:

$$K_{IFS}(A,\ B) = \frac{C_{IFS}(A,\ B)}{\sqrt{E_{IFS}(A)\ .\ E_{IFS}(B)}}$$

Step 4. Rank all the alternatives $A_i (i = 1, 2, \ldots, m)$ and select the most desirable one(s).

VII. Numerical Illustration

An investment enterprise wants to invest some money into a company, where there are five possible companies A_1, A_2, A_3, A_4, and A_5 as alternatives. In order to make a reasonable decision, the investment enterprise invited three experts D_1, D_2, and D_3 to evaluate the alternatives with respect to four attributes G_1, G_2,G_3, and G_4, where G_1 denotes the risk analysis, G_2 denotes the growth analysis, G_3 denotes the social-political impact analysis, and G_4 denotes the environmental impact analysis. The five possible alternatives A_i for i = 1, 2, 3, …., 5 are to be evaluated using the intuitionistic fuzzy numbers, whose weighting vector by the decision makers under the above four attributes is derived by using singular perturbation problem. The three decision makers have reached a consensus on the weight vector, and have provided the weight information in the form of a singularly perturbed differential equation of convection diffusion type, which was solved in section 6. The decision makers want the singular perturbation problem to be solved numerically and the weights derived. Following the instructions of the decision makers, the weight information is derived as $\omega = (0.33305, 0.33343, 0.33351)^T$ and $w = (0.20452, 0.29550, 0.26279, 0.23731)$. The decision maker evaluates the alternatives in terms of the IFS with respect to the attributes to form their decision matrices $R^{(k)} = (r_{ij}^{(k)})_{5x4} = (\mu_{ij}^{(k)}, \gamma_{ij}^{(k)})_{5x4}$ (k=1,2,3) as follows:

$$D_1 = \begin{pmatrix} (0.5,\ 0.4) & (0.5,\ 0.3) & (0.2,\ 0.6) & (0.4,\ 0.4) \\ (0.7,\ 0.3) & (0.7,\ 0.3) & (0.6,\ 0.2) & (0.6,\ 0.2) \\ (0.5,\ 0.4) & (0.6,\ 0.4) & (0.6,\ 0.2) & (0.5,\ 0.3) \\ (0.8,\ 0.2) & (0.7,\ 0.2) & (0.4,\ 0.2) & (0.5,\ 0.2) \\ (0.4,\ 0.3) & (0.4,\ 0.2) & (0.4,\ 0.5) & (0.4,\ 0.6) \end{pmatrix}$$

$$D_2 = \begin{pmatrix} (0.4,\ 0.5) & (0.6,\ 0.2) & (0.5,\ 0.4) & (0.5,\ 0.3) \\ (0.5,\ 0.4) & (0.6,\ 0.2) & (0.6,\ 0.3) & (0.7,\ 0.3) \\ (0.4,\ 0.5) & (0.3,\ 0.5) & (0.4,\ 0.4) & (0.2,\ 0.6) \\ (0.5,\ 0.4) & (0.7,\ 0.2) & (0.4,\ 0.4) & (0.6,\ 0.2) \\ (0.6,\ 0.3) & (0.7,\ 0.2) & (0.4,\ 0.2) & (0.7,\ 0.2) \end{pmatrix}$$

$$D_3 = \begin{pmatrix} (0.4,\ 0.2) & (0.5,\ 0.2) & (0.5,\ 0.3) & (0.5,\ 0.2) \\ (0.5,\ 0.3) & (0.5,\ 0.3) & (0.6,\ 0.2) & (0.7,\ 0.2) \\ (0.4,\ 0.4) & (0.3,\ 0.4) & (0.4,\ 0.3) & (0.3,\ 0.3) \\ (0.5,\ 0.3) & (0.5,\ 0.3) & (0.3,\ 0.5) & (0.5,\ 0.2) \\ (0.6,\ 0.2) & (0.6,\ 0.4) & (0.4,\ 0.4) & (0.6,\ 0.3) \end{pmatrix}$$ By

using the proposed algorithm we obtain:

$$K_{IFS}(\tilde{r}_i^{(1)}, \tilde{r}^+) = 0.73941;\ K_{IFS}(\tilde{r}_i^{(2)}, \tilde{r}^+) = 0.51325;$$

$$K_{IFS}(\tilde{r}_i^{(3)}, \tilde{r}^+) = 0.81283;\ K_{IFS}(\tilde{r}_i^{(4)}, \tilde{r}^+) = 0.53512;$$

$$K_{IFS}(\tilde{r}_i^{(5)}, \tilde{r}^+) = 0.60321.$$

.Rank all the alternatives $A_i, (i = 1, 2, 3, 4, 5)$:

$$A_3 > A_1 > A_5 > A_4 > A_2.$$

Hence, the best alternative is A_3.

VIII. Conclusion

In this work, our aim is to determine the weights of decision makers in which the decision maker weights are completely unknown. So we can present a new approach for finding weights of decision makers in group decision environment based on singular perturbation problem given by the decision maker was proposed. To derive the decision maker weights, the exact and numerical solution of singularly perturbed differential equation of convection-diffusion type problem is considered and its applied in MAGDM problems under intuitionistic fuzzy environment. A newly proposed correlation coefficient of IFS is used as a ranking tool for choosing desirable alternative. Finally, the numerical solutions were decomposed, and decision maker's weights for alternatives were derived and corresponding decision making algorithm was proposed. The proposed method in this paper can relieve a difficult situation in the decision making process especially when the decision maker provides the weight information in the form which is unknown to the stakeholder.

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