International Journal for Research in Engineering Application & Management (IJREAM)
ISSN : 2454-9150 Vol-04, Issue-07, Oct 2018

Stability of a Additive-Quartic Functional Equation

[1*]V. Govindan, Department of Mathematics,Sri Vidya Mandir Arts and Science College,Uthangarai TamilNadu, India-636 902, govindoviya@gmail.com.
[2]Sandra Pinelas, Departmento de Ciências Exatas e Engenharia, Academia Militar, Portugal. sandra.pinelas@gmail.com.
[3]S. Murthy, Department of Mathematics,Government Arts College For Men ,Krishnagiri, Tamil Nadu, India-635 001, smurthy07@yahoo.co.in.
[4]M. Sree Shanmuga Velan, Department of Mathematics,Hosur Institute of Technology and Science Krishnagiri, Tamil Nadu, India-635 115, sreevelanmsc@gmail.com.

Abstract In this paper, the authors investigate and introduce the general solution and generalized Ulam-Hyers Stability of 3-dimensional additive-quartic functional equation of the form

$$f\left(rx_1+r^2x_2+r^3x_3\right)+f\left(-rx_1+r^2x_2+r^3x_3\right)+f\left(rx_1-r^2x_2+r^3x_3\right)+f\left(rx_1+r^2x_2-r^3x_3\right)=2\Big[f\left(rx_1+r^2x_2\right)$$
$$+f\left(r^2x_2+r^3x_3\right)+f\left(rx_1+r^3x_3\right)+f\left(rx_1-r^2x_2\right)+f\left(r^2x_2-r^3x_3\right)+f\left(r^3x_3-rx_1\right)\Big]-2[r^4\left(f\left(x_1\right)+f\left(-x_1\right)\right]$$
$$+r^8\left(f\left(x_2\right)+f\left(-x_2\right)\right)+r^{12}\left(f\left(x_3\right)+f\left(-x_3\right)\right)\Big]-\Big[r\left(f\left(x_1\right)-f\left(-x_1\right)\right)+r^2\left(f\left(x_2\right)-f\left(-x_2\right)\right)+r^3\left(f\left(x_3\right)-f\left(-x_3\right)\right)\Big]$$

in Banach space and Banach algebra using Direct and Fixed point methods.

Keywords — *Additive functional equation, Quartic functional equation, Ulam-Hyers stability, Banach space, fixed point.*
MSC: 39B52, 32B72, 32B82.

I. Introduction

The stability of functional equation was raised by S. M. Ulam[12] for what metric group G is it true that a $\varepsilon-$automorphism of G is necessary near to a strict automorphism?. In 1941, D. H. Hyers[4] gave a positive answer to the question of Ulam for Banach spaces.

The functional equation

$$f\left(x+y\right)=f\left(x\right)+f\left(y\right) \qquad (1.1)$$

is called the Cauchy additive functional equation and it is the most famous functional equation. Since $f\left(x\right)=x$ is the solution of the functional equation (1.1).

Then the functional equation

$$f\left(2x+y\right)+f\left(2x-y\right)=4f\left(x+y\right)+4f\left(x-y\right)+24f\left(x\right)-6f\left(y\right) \qquad (1.2)$$

is called the Quartic functional equation. It is easy to show that the function $f\left(x\right)=x^4$ satisfies the functional equation (1.2), which is called a quartic functional equation and every solution of the quartic functional equation is said to be a quartic function.

The solution and stability of the functional equation for additive and quartic functional equations are discussed in [3,5,6,7,8]. V. Govindan [2], established the general solution for the quadratic functional equation and investigate the stability in Banach spaces. Some of the functional papers are used to develop this paper which are [1,9,10,11].

In this paper, the authors examine the general solution and stability for the Additive-Quartic functional equation is of the form

$$f\left(rx_1+r^2x_2+r^3x_3\right)+f\left(-rx_1+r^2x_2+r^3x_3\right)+f\left(rx_1-r^2x_2+r^3x_3\right)$$
$$+f\left(rx_1+r^2x_2-r^3x_3\right)=2\Big[f\left(rx_1+r^2x_2\right)+f\left(r^2x_2+r^3x_3\right)$$
$$+f\left(rx_1+r^3x_3\right)+f\left(rx_1-r^2x_2\right)+f\left(r^2x_2-r^3x_3\right)+f\left(rx_1-r^3x_3\right)\Big]$$
$$-2[r^4\left(f\left(x_1\right)+f\left(-x_1\right)\right]+r^8\left(f\left(x_2\right)+f\left(-x_2\right)\right)+r^{12}\left(f\left(x_3\right)+f\left(-x_3\right)\right)\Big]$$
$$-\Big[r\left(f\left(x_1\right)-f\left(-x_1\right)\right)+r^2\left(f\left(x_2\right)-f\left(-x_2\right)\right)+r^3\left(f\left(x_3\right)-f\left(-x_3\right)\right)\Big] \qquad (1.3)$$

in Banach space using direct and fixed point method.

Theorem A. (Banach Contraction Principle) Let $\left(X,d\right)$ be a complete metric space and consider a mapping $T:X\to X$ which is strictly contractive mapping, that is

(A1) $d\left(T_X,T_y\right)\le Ld\left(x,y\right)$ for some (Lipschitz constant) $L<1$, then

(i) The mapping T has one and only fixed point $x^*=T\left(x^*\right)$;

(ii) The fixed point for each given element x^* is globally attractive that is

(A2) $\lim_{n\to\infty}T^nx=x^*$, for any starting point $x\in X$;

(iii) One has the following estimation inequalities:

(A3) $d\left(T^nx,x^*\right)\le\frac{1}{1-L}d\left(T^nx,T^{n+1}x\right)$, for all $n\ge0,\ x\in X$.

(A4) $d\left(x,x^*\right)\le\frac{1}{1-L}d\left(x,x^*\right),\quad\forall\,x\in X$.

 DOI : 10.18231/2454-9150.2018.0899

Theorem B. (The alternative of fixed point) Suppose that for a complete generalized metric space (X,d) and a strictly contractive mapping $T : X \to Y$ with Lipschitz constant L. Then, for each given element $x \in X$, either

(B1) $d\left(T^n x, T^{n+1}x\right) = \infty,\ \forall\ n \ge 0$ or

(B2) there exists natural number n_0 such that:

(i) $d\left(T^n x, T^{n+1}x\right) < \infty$ for all $n \ge n_0$

(ii) The sequence $\left(T^n x\right)$ is convergent to a fixed point y^* of T

(iii) y^* is the unique fixed point of T in the set $Y = \left\{ y \in X : d\left(T^n x, y\right) < \infty \right\}$;

$$d\left(y^*, y\right) \le \frac{1}{1-L} d\left(y, Ty\right) \ \forall \ y \in L.$$

II. Solution of the Functional Equation (1.3)

In this section, the authors present the general solution of the functional equation (1.3).

Lemma 2.1 Let X and Y be a real vector spaces. The mapping $f : X \to Y$ satisfies the functional equation (1.3) for all $x_1, x_2, x_3 \in X$, then $f : X \to Y$ satisfies the functional equation (1.1) for all $x, y \in X$.

Theorem 2.2 Let X and Y be a real vector spaces. The mapping $f : X \to Y$ satisfies the functional equation (1.3) for all $x_1, x_2, x_3 \in X$, then $f : X \to Y$ satisfies the functional equation (1.2) for all $x, y \in X$.

Proof: Assume that f satisfies the functional equation (1.2). Let $x = y = 0$ in (1.2), we get $f(0) = 0$. Switching $x = 0$ in (1.2), we have $f(y) = f(-y)$ for all $y \in X$. Replacing $y = 0$ and $y = x$ in (1.2), we arrive

$$f(2x) = 16 f(x)$$

and for all $x \in X$, we have

$$f(3x) = 81 f(x)$$

respectively. In this manner, we generalized to get

$$f(nx) = n^4 f(x)$$

for all $x \in X$ and all $n \in N$. Letting x and y by $rx_1 + r^2x_2$ and $rx_1 - r^2x_2$ in (1.2), respectively, we have

$$f\left(3rx_1 + r^2x_2\right) + f\left(rx_1 + 3r^2x_2\right) = 64 f\left(rx_1\right) + 64 f\left(r^2x_2\right) + 24 f\left(rx_1 + r^2x_2\right) - 6 f\left(rx_1 - r^2x_2\right) \quad (2.1)$$

for all $x_1, x_2 \in X$. Switching rx_1 and r^2x_2 by $rx_1 + r^2x_2$ and $2r^2x_2$ in (1.2), respectively, we get

$$4 f\left(rx_1 + 2r^2x_2\right) + 4 f\left(rx_1\right) = f\left(rx_1 + 3r^2x_2\right) + f\left(rx_1 - r^2x_2\right) + 6 f\left(rx_1 + r^2x_2\right) - 24 f\left(r^2x_2\right) \quad (2.2)$$

for all $x_1, x_2 \in X$. Interchanging rx_1 and r^2x_2 in (2.2), we reach

$$4 f\left(r^2x_2 + 2rx_1\right) + 4 f\left(r^2x_2\right) = f\left(r^2x_2 + 3rx_1\right) + f\left(r^2x_2 - rx_1\right) + 6 f\left(r^2x_2 + rx_1\right) - 24 f\left(rx_1\right) \quad (2.3)$$

for all $x_1, x_2 \in X$. Adding (2.2) and (2.3) and using (2.1), we obtain

$$4 f\left(rx_1 + 2r^2x_2\right) + 4 f\left(rx_1\right) + 4 f\left(r^2x_2 + 2rx_1\right) + 4 f\left(r^2x_2\right)$$
$$= f\left(rx_1 + 3r^2x_2\right) + f\left(3rx_1 + r^2x_2\right) + 12 f\left(rx_1 + r^2x_2\right)$$
$$+ f\left(rx_1 - r^2x_2\right) + f\left(r^2x_2 - rx_1\right) - 24 f\left(rx_1\right) - 24 f\left(r^2x_2\right)$$

for all $x_1, x_2 \in X$. Now, using the result (2.1), we get

$$4 f\left(rx_1 + 2r^2x_2\right) + 4 f\left(r^2x_2 + 2rx_1\right) = 64 f\left(rx_1\right) + 64 f\left(r^2x_2\right)$$
$$+24 f\left(rx_1 + r^2x_2\right) - 6 f\left(rx_1 - r^2x_2\right) + 12 f\left(rx_1 + r^2x_2\right)$$
$$f\left(rx_1 - r^2x_2\right) + f\left(r^2x_2 - rx_1\right) - 28 f\left(rx_1\right) - 28 f\left(r^2x_2\right)$$

for all $x_1, x_2 \in X$. Again Using $f(-x) = f(x)$, we reach

$$f\left(rx_1 + 2r^2x_2\right) + f\left(r^2x_2 + 2rx_1\right) = 9 f\left(rx_1\right) + 9 f\left(r^2x_2\right) + 9 f\left(rx_1 + r^2x_2\right) - f\left(rx_1 - r^2x_2\right) \quad (2.4)$$

for all $x_1, x_2 \in X$. Switching $x = rx_1; y = r^3x_3$ in (1.2), we have

$$f\left(2rx_1 + r^3x_3\right) + f\left(2rx_1 - r^3x_3\right) = 24 f\left(rx_1\right) - 6 f\left(r^3x_3\right) + 4 f\left(rx_1 + r^3x_3\right) + 4 f\left(rx_1 - r^3x_3\right) \quad (2.5)$$

for all $x_1, x_3 \in X$. Replacing $x = r^2x_2$ and $y = r^3x_3$ in (1.2), we get

$$f\left(2r^2x_2 + r^3x_3\right) + f\left(2r^2x_2 - r^3x_3\right) = 24 f\left(r^2x_2\right) - 6 f\left(r^3x_3\right) + 4 f\left(r^2x_2 + r^3x_3\right) + 4 f\left(r^2x_2 - r^3x_3\right) \quad (2.6)$$

for all $x_2, x_3 \in X$. Adding (2.5) and (2.6), we arrive

$$9 f\left(2rx_1 + r^3x_3\right) + 9 f\left(2rx_1 - r^3x_3\right) + 9 f\left(2r^2x_2 + r^3x_3\right)$$
$$+9 f\left(2r^2x_2 - r^3x_3\right) = 36 f\left(rx_1 + r^3x_3\right) + 36 f\left(rx_1 - r^3x_3\right)$$
$$+36 f\left(r^2x_2 + r^3x_3\right) + 36 f\left(r^2x_2 - r^3x_3\right) + 216 f\left(rx_1\right)$$
$$+216 f\left(r^2x_2\right) - 108 f\left(r^3x_3\right) \quad (2.7)$$

 DOI : 10.18231/2454-9150.2018.0899

ISSN : 2454-9150 Vol-04, Issue-07, Oct 2018

for all $x_1, x_2, x_3 \in X$. Interchanging $rx_1 = 2rx_1 + r^3x_3$ and $r^2x_2 = 2r^2x_2 + r^3x_3$ in the equation (2.4), we attain

$$\begin{aligned} & f\left(2rx_1 + 4r^2x_2 + 3r^3x_3\right) + f\left(4rx_1 + 2r^2x_2 + 3r^3x_3\right) \\ & = 9f\left(2rx_1 + r^3x_3\right) + 9f\left(2r^2x_2 + r^3x_3\right) \\ & \quad + 9f\left(2rx_1 + 2r^2x_2 + 2r^3x_3\right) - f\left(2rx_1 - 2r^2x_2\right) \end{aligned} \tag{2.8}$$

for all $x_1, x_2, x_3 \in X$. Replacing rx_1 and r^2x_2 by $2rx_1 - r^3x_3$ and $r^2x_2 = 2r^2x_2 - r^3x_3$ in (2.4), we reach

$$\begin{aligned} & f\left(2rx_1 + 4r^2x_2 - 3r^3x_3\right) + f\left(4rx_1 + 2r^2x_2 - 3r^3x_3\right) \\ & = 9f\left(2rx_1 - r^3x_3\right) + 9f\left(2r^2x_2 - r^3x_3\right) \\ & \quad + 9f\left(2rx_1 + 2r^2x_2 - 2r^3x_3\right) - f\left(2rx_1 - 2r^2x_2\right) \end{aligned} \tag{2.9}$$

for all $x_1, x_2, x_3 \in X$. Adding (2.8) and (2.9) and also using (1.2), we get

$$\begin{aligned} & 9f\left(2rx_1 + r^3x_3\right) + 9f\left(2r^2x_2 + r^3x_3\right) + 9f\left(2rx_1 - r^3x_3\right) \\ & + 9f\left(2r^2x_2 - r^3x_3\right) = 4f\left(rx_1 + 2r^2x_2 + 3r^3x_3\right) \\ & + 4f\left(rx_1 + 2r^2x_2 - 3r^3x_3\right) + 24f\left(rx_1 + 2r^2x_2\right) - 6f\left(3r^3x_3\right) \\ & + 4f\left(2rx_1 + r^2x_2 + 3r^3x_3\right) + 4f\left(2rx_1 + r^2x_2 - 3r^3x_3\right) \\ & + 24f\left(2rx_1 + r^2x_2\right) - 6f\left(3r^3x_3\right) - 144f\left(rx_1 + r^2x_2 + r^3x_3\right) \\ & - 144f\left(rx_1 + r^2x_2 - r^3x_3\right) + 32f\left(rx_1 - r^2x_2\right) \end{aligned} \tag{2.10}$$

for all $x_1, x_2, x_3 \in X$. By the equations (2.7) and (2.10), we attain

$$\begin{aligned} & 36f\left(rx_1 + r^3x_3\right) + 36f\left(rx_1 - r^3x_3\right) + 216f\left(rx_1\right) \\ & - 54f\left(r^3x_3\right) + 36f\left(r^2x_2 + r^3x_3\right) + 36f\left(r^2x_2 - r^3x_3\right) \\ & + 216f\left(r^2x_2\right) - 54f\left(r^3x_3\right) = 4f\left(rx_1 + 2r^2x_2 + 3r^3x_3\right) \\ & + 4f\left(rx_1 + 2r^2x_2 - 3r^3x_3\right) + 24f\left(rx_1 + 2r^2x_2\right) - 6f\left(3r^3x_3\right) \\ & + 4f\left(2rx_1 + r^2x_2 + 3r^3x_3\right) + 4f\left(2rx_1 + r^2x_2 - 3r^3x_3\right) \\ & + 24f\left(2rx_1 + r^2x_2\right) - 6f\left(3r^3x_3\right) - 144f\left(rx_1 + r^2x_2 + r^3x_3\right) \\ & - 144f\left(rx_1 + r^2x_2 - r^3x_3\right) + 32f\left(rx_1 - r^2x_2\right) \end{aligned} \tag{2.11}$$

for all $x_1, x_2, x_3 \in X$. Replacing rx_1 and r^2x_2 by $2rx_1 + r^3x_3$ and $r^2x_2 = 2r^2x_2 - r^3x_3$ in (2.4), we arrive

$$\begin{aligned} & f\left(2rx_1 + 4r^2x_2 - r^3x_3\right) + f\left(4rx_1 + 2r^2x_2 + r^3x_3\right) = 9f\left(2rx_1 + r^3x_3\right) \\ & + 9f\left(2r^2x_2 - r^3x_3\right) + 9f\left(2rx_1 + 2r^2x_2\right) - f\left(2rx_1 - 2r^2x_2 + 2r^3x_3\right) \end{aligned}$$

which implies that

$$\begin{aligned} & 9f\left(2rx_1 + r^3x_3\right) + 9f\left(2r^2x_2 - r^3x_3\right) = f\left(2rx_1 + 4r^2x_2 - r^3x_3\right) \\ & + f\left(4rx_1 + 2r^2x_2 + r^3x_3\right) - 9f\left(2rx_1 + 2r^2x_2\right) + f\left(2rx_1 - 2r^2x_2 + 2r^3x_3\right) \end{aligned} \tag{2.12}$$

for all $x_1, x_2, x_3 \in X$. Replacing $rx_1 = 2rx_1 - r^3x_3$ and $r^2x_2 = 2r^2x_2 + r^3x_3$ in (2.4), we have

$$\begin{aligned} & 9f\left(2rx_1 - r^3x_3\right) + 9f\left(2r^2x_2 + r^3x_3\right) \\ & = f\left(2rx_1 + 4r^2x_2 + r^3x_3\right) + f\left(4rx_1 + 2r^2x_2 - r^3x_3\right) \\ & \quad - 9f\left(2rx_1 + 2r^2x_2\right) + f\left(2rx_1 - 2r^2x_2 - 2r^3x_3\right) \end{aligned} \tag{2.13}$$

for all $x_1, x_2, x_3 \in X$. Adding (2.12) and (2.13), we arrive

$$\begin{aligned} & 9f\left(2rx_1 + r^3x_3\right) + 9f\left(2r^2x_2 - r^3x_3\right) + 9f\left(2rx_1 - r^3x_3\right) \\ & + 9f\left(2r^2x_2 + r^3x_3\right) = f\left(2rx_1 + 4r^2x_2 - r^3x_3\right) + f\left(4rx_1 + 2r^2x_2 + r^3x_3\right) \\ & + f\left(2rx_1 + 4r^2x_2 + r^3x_3\right) + f\left(4rx_1 + 2r^2x_2 - r^3x_3\right) - 9f\left(2rx_1 + 2r^2x_2\right) \\ & - 9f\left(2rx_1 + 2r^2x_2\right) + f\left(2rx_1 - 2r^2x_2 + 2r^3x_3\right) + f\left(2rx_1 - 2r^2x_2 - 2r^3x_3\right) \end{aligned} \tag{2.14}$$

for all $x_1, x_2, x_3 \in X$. Using (1.2) in the above equation, we obtain

$$\begin{aligned} & 9f\left(2rx_1 + r^3x_3\right) + 9f\left(2r^2x_2 - r^3x_3\right) + 9f\left(2rx_1 - r^3x_3\right) + 9f\left(2r^2x_2 + r^3x_3\right) \\ & = 4f\left(rx_1 + 2r^2x_2 + r^3x_3\right) + 4f\left(rx_1 + 2r^2x_2 - r^3x_3\right) + 24f\left(rx_1 + 2r^2x_2\right) \\ & - 6f\left(r^3x_3\right) + 4f\left(2rx_1 + r^2x_2 + r^3x_3\right) + 4f\left(2rx_1 + r^2x_2 - r^3x_3\right) \\ & + 24f\left(2rx_1 + r^2x_2\right) - 6f\left(r^3x_3\right) - 288f\left(rx_1 + r^2x_2\right) + 16f\left(rx_1 - r^2x_2 + r^3x_3\right) \\ & + 16f\left(rx_1 - r^2x_2 - r^3x_3\right) \end{aligned} \tag{2.15}$$

for all $x_1, x_2, x_3 \in X$. From (2.15), we have

$$\begin{aligned} & 9f\left(2rx_1 + r^3x_3\right) + 9f\left(2r^2x_2 - r^3x_3\right) + 9f\left(2rx_1 - r^3x_3\right) + 9f\left(2r^2x_2 + r^3x_3\right) \\ & = 4f\left(rx_1 + 2r^2x_2 + r^3x_3\right) + 4f\left(rx_1 + 2r^2x_2 - r^3x_3\right) + 24f\left(rx_1 + 2r^2x_2\right) \\ & - 6f\left(r^3x_3\right) + 4f\left(2rx_1 + r^2x_2 + r^3x_3\right) + 4f\left(2rx_1 + r^2x_2 - r^3x_3\right) \\ & + 24f\left(2rx_1 + r^2x_2\right) - 6f\left(r^3x_3\right) - 288f\left(rx_1 + r^2x_2\right) \\ & + 16f\left(rx_1 - r^2x_2 + r^3x_3\right) + 16f\left(rx_1 - r^2x_2 - r^3x_3\right) \end{aligned} \tag{2.16}$$

for all $x_1, x_2, x_3 \in X$. Switching r^3x_3 by $3r^3x_3$ in (2.16), we get

$$\begin{aligned} & 9f\left(2rx_1 + 3r^3x_3\right) + 9f\left(2r^2x_2 - 3r^3x_3\right) + 9f\left(2rx_1 - 3r^3x_3\right) \\ & + 9f\left(2r^2x_2 + 3r^3x_3\right) = 4f\left(rx_1 + 2r^2x_2 + 3r^3x_3\right) \\ & + 4f\left(rx_1 + 2r^2x_2 - 3r^3x_3\right) + 24f\left(rx_1 + 2r^2x_2\right) \\ & - 6f\left(3r^3x_3\right) + 4f\left(2rx_1 + r^2x_2 + 3r^3x_3\right) \\ & + 4f\left(2rx_1 + r^2x_2 - 3r^3x_3\right) + 24f\left(2rx_1 + r^2x_2\right) \\ & - 6f\left(3r^3x_3\right) - 288f\left(rx_1 + r^2x_2\right) + 16f\left(rx_1 - r^2x_2 + 3r^3x_3\right) \\ & + 16f\left(rx_1 - r^2x_2 - 3r^3x_3\right) \end{aligned} \tag{2.17}$$

for all $x_1, x_2, x_3 \in X$. Using (2.11) in (2.17), we have

 DOI : 10.18231/2454-9150.2018.0899

$$9f\left(2rx_1+r^3x_3\right)+9f\left(2r^2x_2-r^3x_3\right)+9f\left(2rx_1-r^3x_3\right)$$
$$+9f\left(2r^2x_2+r^3x_3\right)=36f\left(rx_1+r^3x_3\right)+36f\left(rx_1-r^3x_3\right)$$
$$+216f\left(rx_1\right)-54f\left(r^3x_3\right)+36f\left(r^2x_2+r^3x_3\right)$$
$$+36f\left(r^2x_2-r^3x_3\right)+216f\left(r^2x_2\right)-54f\left(r^3x_3\right)$$
$$+144f\left(rx_1+r^2x_2+r^3x_3\right)+144f\left(rx_1+r^2x_2-r^3x_3\right)$$
$$-32f\left(x_1-r^2x_2\right)-288f\left(x_1+r^2x_2\right)+16f\left(rx_1-r^2x_2+3r^3x_3\right)$$
$$+16f\left(rx_1-r^2x_2-3r^3x_3\right) \tag{2.18}$$

for all $x_1, x_2, x_3 \in X$. Replacing $rx_1 = rx_1 - r^2x_2 + 3r^3x_3$ and $r^2x_2 = rx_1 - r^2x_2 - 3r^3x_3$ in (2.4), we have

$$9f\left(rx_1-r^2x_2+3r^3x_3\right)+9f\left(rx_1-r^2x_2-3r^3x_3\right)=81f\left(rx_1-r^2x_2-r^3x_3\right)$$
$$+81f\left(rx_1-r^2x_2+r^3x_3\right)-144f\left(rx_1-r^2x_2\right)+1296f\left(r^3x_3\right) \tag{2.19}$$

for all $x_1, x_2, x_3 \in X$. Divided by $\left(\frac{16}{9}\right)$, we get

$$16f\left(rx_1-r^2x_2+3r^3x_3\right)+16f\left(rx_1-r^2x_2-3r^3x_3\right)=144f\left(rx_1-r^2x_2-r^3x_3\right)$$
$$+144f\left(rx_1-r^2x_2+r^3x_3\right)-256f\left(rx_1-r^3x_3\right)+2304f\left(r^3x_3\right) \tag{2.20}$$

for all $x_1, x_2, x_3 \in X$. Substitute (2.20) in (2.18), we receive

$$9f\left(2rx_1+3r^3x_3\right)+9f\left(2r^2x_2-3r^3x_3\right)+9f\left(2rx_1-3r^3x_3\right)$$
$$+9f\left(2r^2x_2+3r^3x_3\right)=36f\left(rx_1+r^3x_3\right)+36f\left(rx_1-r^3x_3\right)$$
$$+216f\left(rx_1\right)-54f\left(r^3x_3\right)+36f\left(r^2x_2+r^3x_3\right)+36f\left(r^2x_2-r^3x_3\right)$$
$$+216f\left(r^2x_2\right)-54f\left(r^3x_3\right)+144f\left(rx_1+r^2x_2+r^3x_3\right)$$
$$+144f\left(rx_1+r^2x_2-r^3x_3\right)-32f\left(rx_1-r^2x_2\right)-288f\left(x_1+r^2x_2\right)$$
$$+144f\left(rx_1-r^2x_2-r^3x_3\right)+144f\left(rx_1-r^2x_2+r^3x_3\right)$$
$$+2304f\left(r^3x_3\right) \tag{2.21}$$

for all $x_1, x_2, x_3 \in X$. Interchanging $rx_1 = 2rx_1 + 3r^3x_3$ *and* $r^2x_2 = 2rx_1 - 3r^3x_3$ in (2.4), we reach

$$f\left(6rx_1-3r^3x_3\right)+f\left(6rx_1+3r^3x_3\right)=9f\left(2rx_1+3r^3x_3\right)+9f\left(2rx_1-3r^3x_3\right)$$
$$+9f\left(4rx_1\right)-f\left(6r^3x_3\right) \tag{2.22}$$

for all $x_1, x_2, x_3 \in X$. Replacing $rx_1 = 2r^2x_2 - 3r^3x_3$ *and* $r^2x_2 = 2r^2x_2 + 3r^3x_3$ in (2.4), we receive

$$f\left(6r^2x_2+3r^3x_3\right)+f\left(6r^2x_2-3r^3x_3\right)=9f\left(2r^2x_2-3r^3x_3\right)$$
$$+9f\left(2r^2x_2+3r^3x_3\right)+9f\left(4r^2x_2\right)-f\left(6r^3x_3\right) \tag{2.23}$$

for all $x_1, x_2, x_3 \in X$. Adding (2.23) and (2.24), we achieve

$$9f\left(2rx_1+3r^3x_3\right)+9f\left(2r^2x_2-3r^3x_3\right)+9f\left(2r^2x_2-3r^3x_3\right)$$
$$+9f\left(2r^2x_2+3r^3x_3\right)=324f\left(rx_1+r^3x_3\right)+324f\left(rx_1-r^3x_3\right)$$
$$+1944f\left(rx_1\right)-486f\left(r^3x_3\right)+324f\left(r^2x_2+r^3x_3\right)$$
$$+324f\left(r^2x_2-r^3x_3\right)+1944f\left(r^2x_2\right)-486f\left(r^3x_3\right)$$
$$-2304f\left(rx_1\right)-2304f\left(r^2x_2\right)-2592f\left(r^3x_3\right) \tag{2.24}$$

for all $x_1, x_2, x_3 \in X$. From (2.21) and (2.24), L. H. S. are equal, we receive

$$36f\left(rx_1+r^3x_3\right)+36f\left(r^2x_2-r^3x_3\right)+216f\left(rx_1\right)-54f\left(r^3x_3\right)$$
$$+36f\left(r^2x_2+r^3x_3\right)+36f\left(r^2x_2-r^3x_3\right)+216f\left(r^2x_2\right)$$
$$-54f\left(r^3x_3\right)+144f\left(rx_1+r^2x_2+r^3x_3\right)+144f\left(rx_1+r^2x_2-r^3x_3\right)$$
$$-32f\left(rx_1-r^2x_2\right)-288f\left(rx_1+r^2x_2\right)+144f\left(rx_1-r^2x_2-r^3x_3\right)$$
$$+144f\left(rx_1-r^2x_2+r^3x_3\right)-256f\left(rx_1-r^2x_2\right)+2304f\left(r^3x_3\right)$$
$$=324f\left(rx_1+r^3x_3\right)+324f\left(rx_1-r^3x_3\right)+1944f\left(rx_1\right)-486f\left(r^3x_3\right)$$
$$+324f\left(r^2x_2+r^3x_3\right)+324f\left(r^2x_2-r^3x_3\right)+1944f\left(r^2x_2\right)$$
$$-486f\left(r^3x_3\right)-2304f\left(rx_1\right)-2304f\left(r^2x_2\right)+2592f\left(r^3x_3\right) \tag{2.25}$$

for all $x_1, x_2, x_3 \in X$. From the resultant equation (2.26), we get

$$f\left(rx_1+r^2x_2+r^3x_3\right)+f\left(rx_1+r^2x_2-r^3x_3\right)+f\left(rx_1-r^2x_2-r^3x_3\right)$$
$$+f\left(rx_1+r^2x_2-r^3x_3\right)=2\left(f\left(rx_1+r^2x_2\right)+f\left(rx_1+r^3x_3\right)+f\left(r^2x_2+r^3x_3\right)\right)$$
$$+2\left(f\left(rx_1-r^2x_2\right)+f\left(rx_1-r^3x_3\right)+f\left(r^2x_2-r^3x_3\right)\right)-4\left(f\left(rx_1\right)+f\left(r^2x_2\right)+f\left(r^3x_3\right)\right) \tag{2.26}$$

for all $x_1, x_2, x_3 \in X$. Adding $rf\left(x_1\right)+r^2f\left(x_2\right)+r^3f\left(x_3\right)$ on both sides of (2.26) and using evenness of f, we desired our required result (1.3).

Conversely, assume that $f: X \to Y$ satisfies the functional equation (1.3). Now we prove that the function $f: X \to Y$ satisfies the functional equation (1.2) Now replacing $\left(x_1, x_2, x_3\right)$ by $\left(\frac{x}{r}, \frac{x}{r^2}, \frac{y}{r^3}\right)$, we arrive (1.2). Hence the proof.

In section 3 and 4, we take X be a normed space and Y be a Banach space. For notational handiness, we define a function $P: X \to Y$ by

DOI : 10.18231/2454-9150.2018.0899

$$P\left(x_1,x_2,x_3\right)=f\left(rx_1+r^2x_2+r^3x_3\right)+f\left(-rx_1+r^2x_2+r^3x_3\right)$$
$$+f\left(rx_1-r^2x_2+r^3x_3\right)+f\left(rx_1+r^2x_2-r^3x_3\right)$$
$$-2\left[f\left(rx_1+r^2x_2\right)+f\left(r^2x_2+r^3x_3\right)+f\left(rx_1+r^3x_3\right)\right.$$
$$\left.+f\left(rx_1-r^2x_2\right)+f\left(r^2x_2-r^3x_3\right)+f\left(rx_1-r^3x_3\right)\right]$$
$$+2[r^4\left(f\left(x_1\right)+f\left(-x_1\right)\right)+r^8\left(f\left(x_2\right)+f\left(-x_2\right)\right)$$
$$+r^{12}\left(f\left(x_3\right)+f\left(-x_3\right)\right)\Big]+\Big[r\left(f\left(x_1\right)-f\left(-x_1\right)\right)$$
$$+r^2\left(f\left(x_2\right)-f\left(-x_2\right)\right)+r^3\left(f\left(x_3\right)-f\left(-x_3\right)\right)\Big]$$

for all $x_1,x_2,x_3\in X$.

III. Stability of 3-Dimensional Functional Equation (1.3) – Direct Method

In this section, the authors discussed the generalized Ulam-Hyers stability of 3-dimensional functional equation (1.3) in Banach space using Direct Method.

Lemma 3.1 Let $j\in\{-1,1\}$. Let $\alpha: X^3\to[0,\infty)$ be a function such that

$$\lim_{k\to\infty}\frac{\alpha\left(r^{kj}x_1,r^{kj}x_2,r^{kj}x_3\right)}{r^{kj}}=0$$

for all $x_1,x_2,x_3\in X$ and let $P: X\to Y$ be a function satisfying the inequality

$$\left\|P\left(x_1,x_2,x_3\right)\right\|\le\alpha\left(x_1,x_2,x_3\right)$$

for all $x_1,x_2,x_3\in X$. Then there exists a unique additive function $A: X\to Y$ such that

$$\left\|f\left(x\right)-A\left(x\right)\right\|\le\frac{1}{2r}\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\mu\left(r^kx\right)}{r^k}$$

where $\mu\left(x\right)=\phi\left(x,0,0\right)$

for all $x\in X$. The mapping $A\left(x\right)$ is defined by

$$A\left(x\right)=\lim_{l\to\infty}\frac{f\left(r^lx\right)}{r^l}$$

for all $x\in X$.

The following corollary is an immediate consequence of the Lemma 3.1 concerning the stability of (1.3).

Corollary 3.2 Let ε and s be a non-negative real numbers. If a function $P: X\to Y$ satisfying the inequality

$$\left\|P\left(x_1,x_2,x_3\right)\right\|\le\begin{cases}\varepsilon,\\ \varepsilon\left\{\sum_{i=1}^{3}\|x_i\|^s\right\},\\ \varepsilon\left\{\prod_{i=1}^{3}\|x_i\|^s+\sum_{i=1}^{3}\|x_i\|^{3s}\right\},\end{cases}$$

for all $x_1,x_2,x_3\in X$. Then there exists a unique additive function $A: X\to Y$ such that

$$\left\|f\left(x\right)-A\left(x\right)\right\|\le\begin{cases}\frac{\varepsilon}{2|r-1|}\\ \frac{\varepsilon\|x\|^s}{2|r-r^s|} & ;s\ne1\\ \frac{\varepsilon\|x\|^{3s}}{2|r-r^{3s}|} & ;s\ne\frac{1}{3}\end{cases}$$

for all $x\in X$.

Theorem 3.3 Let $j\in\{-1,1\}$. Let $\alpha: X^3\to[0,\infty)$ be a function such that

$$\lim_{k\to\infty}\frac{\alpha\left(r^{kj}x_1,r^{kj}x_2,r^{kj}x_3\right)}{r^{4kj}}=0\tag{3.1}$$

for all $x_1,x_2,x_3\in X$ and let $P: X\to Y$ be a function satisfying the inequality

$$\left\|P\left(x_1,x_2,x_3\right)\right\|\le\alpha\left(x_1,x_2,x_3\right)\tag{3.2}$$

for all $x_1,x_2,x_3\in X$. Then there exists a unique quartic function $Q: X\to Y$ such that

$$\left\|f\left(x\right)-Q\left(x\right)\right\|\le\frac{1}{4r^4}\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\mu\left(r^kx\right)}{r^{4k}}\tag{3.3}$$

$$\text{where }\mu\left(x\right)=\phi\left(x,0,0\right)\tag{3.4}$$

for all $x\in X$. The mapping $Q\left(x\right)$ is defined by

$$Q\left(x\right)=\lim_{l\to\infty}\frac{f\left(r^lx\right)}{r^{4l}}\tag{3.5}$$

for all $x\in X$.

Proof. Assume that $j=1$. Replacing $\left(x_1,x_2,x_3\right)$ by $\left(x,0,0\right)$ in (3.2), we get

$$\left\|4r^4f\left(x\right)-4f\left(rx\right)\right\|\le\alpha\left(x,0,,0\right)\tag{3.6}$$

for all $x\in X$. It follows from (3.6) that

$$\left\|\frac{f\left(rx\right)}{r^4}-f\left(x\right)\right\|\le\frac{1}{4r^4}\alpha\left(x,0,0\right)\tag{3.7}$$

for all $x\in X$. Now replacing x by rx and dividing by r^4 in (3.7), we arrive

$$\left\|\frac{f\left(r^2x\right)}{r^8}-\frac{f\left(rx\right)}{r^4}\right\|\le\frac{1}{4r^8}\alpha\left(rx,0,0\right)\tag{3.8}$$

for all $x\in X$. Adding (3.7) and (3.8), we have

$$\left\|\frac{f\left(r^2x\right)}{r^8}-f\left(x\right)\right\|\le\frac{1}{4r^4}\left[\alpha\left(x,0,0\right)+\frac{\alpha\left(rx,0,0\right)}{r^4}\right]$$

 DOI : 10.18231/2454-9150.2018.0899

for all $x \in X$. In general for any positive integer "l", one can easy to verify that

$$\left\| \frac{f(r^l x)}{r^{4l}} - f(x) \right\| \le \frac{1}{4r^4} \sum_{k=0}^{l-1} \frac{\mu(r^k x)}{r^{4k}}$$

$$\left\| \frac{f(r^l x)}{r^{4l}} - f(x) \right\| \le \frac{1}{4r^4} \sum_{k=0}^{\infty} \frac{\mu(r^k x)}{r^{4k}} \tag{3.9}$$

for all $x \in X$. In order to prove the convergence of the sequence $\left\{ \frac{f(r^l x)}{r^{4l}} \right\}$, replacing x by $r^m x$ and dividing r^{4m} in (3.9), for $l, m > 0$, we get

$$\left\| \frac{f(r^{l+m} x)}{r^{4(l+m)}} - \frac{f(r^m x)}{r^{4m}} \right\| \le \frac{1}{4r^4} \sum_{k=0}^{l-1} \frac{\mu(r^{k+m} x)}{r^{4(k+m)}} \to 0 \ as \ m \to \infty \tag{3.10}$$

for all $x \in X$. Hence the sequence $\left\{ \frac{f(r^l x)}{r^{4l}} \right\}$ is a Cauchy sequence. Since Y is complete, there exists a mapping $Q: X \to Y$ such that

$$Q(x) = \lim_{l \to \infty} \frac{f(r^l x)}{r^{4l}}$$

for all $x \in X$. Letting $l \to \infty$ in (3.9) we see that (3.4) holds for $x \in X$. To prove that Q satisfies (1.3), replacing (x_1, x_2, x_3) by $(r^m x, r^{2m} x, r^{3m} x)$ and dividing r^{4m} in (3.2), we arrive

$$\frac{1}{r^{4m}} \left\| P(r^m x, r^{2m} x, r^{3m} x) \right\| \le \frac{1}{r^{4m}} \alpha(r^m x, r^{2m} x, r^{3m} x)$$

for all $x_1, x_2, x_3 \in X$. Letting $m \to \infty$ in above inequality and using the definition of $Q(x)$, we see that $Q(x_1, x_2, x_3) = 0$. Hence Q satisfies (1.3) for all $x_1, x_2, x_3 \in X$. To show that Q is unique. Let $R(x)$ be the another quartic mapping satisfying (1.3) and (3.4), then

$$\|Q(x) - R(x)\|$$

$$\le \frac{1}{r^{4m}} \left\{ \|Q(r^m x) - f(r^m x)\| + \|f(r^m x) - R(r^m x)\| \right\}$$

$$\le \frac{1}{4r^4} \sum_{k=0}^{\infty} \frac{\mu(r^{k+m} x)}{r^{4(k+m)}} \to 0 \ as \ m \to \infty$$

for all $x \in X$. Hence Q is unique. For $j = -1$, we can prove a similar stability result. This completes the proof of the theorem.

The following corollary is an immediate consequence of the Theorem 3.3 concerning the stability of (1.3).

Corollary 3.2 Let ε and s be a non-negative real numbers. If a function $P: X \to Y$ satisfying the inequality

$$\|P(x_1, x_2, x_3)\| \le \begin{cases} \varepsilon, \\ \varepsilon \left\{ \sum_{i=1}^{3} \|x_i\|^s \right\}, \\ \varepsilon \left\{ \prod_{i=1}^{3} \|x_i\|^s + \sum_{i=1}^{3} \|x_i\|^{3s} \right\}, \end{cases} \tag{3.11}$$

for all $x_1, x_2, x_3 \in X$. Then there exists a unique quartic function $Q: X \to Y$ such that

$$\|f(x) - Q(x)\| \le \begin{cases} \frac{\varepsilon}{4|r^4 - 1|} \\ \frac{\varepsilon \|x\|^s}{4|r^4 - r^s|} & ; s \ne 4 \\ \frac{\varepsilon \|x\|^{3s}}{4|r^4 - r^{3s}|} & ; s \ne \frac{4}{3} \end{cases} \tag{3.12}$$

for all $x \in X$.

IV. Stability of 3-Dimensional Functional Equation (1.3)-Fixed Point Method

In this section, we establish the generalized Ulam-Hyers stability of the 3-dimensional functional equation (1.3) in Banach space with the help of fixed point method.

Lemma 4.1 Let $P: X \to Y$ be a mapping for which there exists a function $\alpha: X^3 \to [0, \infty)$ with the condition

$$\lim_{k \to \infty} \frac{\alpha(\eta_i^k x_1, \eta_i^k x_2, \eta_i^k x_3)}{\eta_i^k} = 0$$

where $\eta_i = \begin{cases} r, & i = 0; \\ \frac{1}{r} & i = 1; \end{cases}$ satisfying the functional inequality

$$\|P(x_1, x_2, x_3)\| \le \alpha(x_1, x_2, x_3)$$

for all $x_1, x_2, x_3 \in X$. If there exists $L = L(i)$ such that the function

$$x \to \gamma(x) = \frac{1}{2} \alpha\left(\frac{x}{r}, 0, 0\right)$$

has the property

$$\frac{\gamma(\eta_i x)}{\eta_i} = L\gamma(x)$$

 DOI : 10.18231/2454-9150.2018.0899

for all $x \in X$. Then there exists a unique additive function $Q: X \to Y$ satisfying the functional equation (1.3) and

$$\left\| f(x) - Q(x) \right\| \le \frac{L^{1-i}}{1-L} \gamma(x)$$

for all $x \in X$.

The following corollary is a immediate consequence of Lemma 4.1concerning the stability of (1.3).

Corollary 4.2 Let ε and s be a non-negative real numbers. If a function $P: X \to Y$ satisfies the inequality

$$\left\| F(x_1, x_2, x_3) \right\| \le \begin{cases} \varepsilon, \\ \varepsilon \left\{ \sum_{i=1}^{3} \left\| x_i \right\|^s \right\}, \\ \varepsilon \left\{ \prod_{i=1}^{3} \left\| x_i \right\|^s + \sum_{i=1}^{3} \left\| x_i \right\|^{3s} \right\}, \end{cases}$$

for all $x_1, x_2, x_3 \in X$. Then there exists an unique additive function such that

$$\left\| f(x) - Q(x) \right\| \le \begin{cases} \dfrac{\varepsilon}{2|r-1|} \\ \dfrac{\varepsilon \left\| x \right\|^s}{2\left| r - r^s \right|} & ; s \ne 1 \\ \dfrac{\varepsilon \left\| x \right\|^{3s}}{2\left| r - r^{3s} \right|} & ; s \ne \dfrac{1}{3} \end{cases}$$

for all $x \in X$.

Theorem 4.3 Let $P: X \to Y$ be a mapping for which there exists a function $\alpha: X^3 \to [0, \infty)$ with the condition

$$\lim_{k \to \infty} \frac{\alpha\left(\eta_i^k x_1, \eta_i^k x_2, \eta_i^k x_3 \right)}{\eta_i^{4k}} = 0 \tag{4.1}$$

where $\eta_i = \begin{cases} r, & i = 0; \\ \frac{1}{r} & i = 1; \end{cases}$ satisfying the functional inequality

$$\left\| P(x_1, x_2, x_3) \right\| \le \alpha(x_1, x_2, x_3) \tag{4.2}$$

for all $x_1, x_2, x_3 \in X$. If there exists $L = L(i)$ such that the function

$$x \to \gamma(x) = \frac{1}{4} \alpha\left(\frac{x}{r}, 0, 0 \right)$$

has the property

$$\frac{\gamma(\eta_i x)}{\eta_i^4} = L\gamma(x) \tag{4.3}$$

for all $x \in X$. Then there exists a unique quartic function $Q: X \to Y$ satisfying the functional equation (1.3) and

$$\left\| f(x) - Q(x) \right\| \le \frac{L^{1-i}}{1-L} \gamma(x) \tag{4.4}$$

for all $x \in X$.

Proof. Let d be a general metric on Ω, such tha

$$d(p,q) = \inf\left\{ k \in (0,\infty) : \left\| p(x) - q(x) \right\| \le k\gamma(x), x \in X \right\}$$

It is easy to see that (Ω, d) is complete. Define $T: \Omega \to \Omega$ by $Tg(x) = \frac{1}{\eta_i^4} g(\eta_i x)$, for all $x \in X$. For $p, q \in \Omega$ and $x \in X$, we have

$$d(p,q) = k \Rightarrow \left\| p(x) - q(x) \right\| \le k\gamma(x),$$

$$\Rightarrow \left\| \frac{p(\eta_i x)}{\eta_i^4} - \frac{q(\eta_i x)}{\eta_i^4} \right\| \le \frac{1}{\eta_i^4} k\gamma(\eta_i x),$$

$$\Rightarrow \left\| Tp(x) - Tq(x) \right\| \le \frac{1}{\eta_i^4} k\gamma(\eta_i x),$$

$$\Rightarrow \left\| Tp(x) - Tq(x) \right\| \le Lk\gamma(x) \Rightarrow d(Tp(x), Tq(x)) \le kL$$

That is $d(Tp, Tq) \le Ld(p,q)$. Therefore T is strictly contractive mapping on Ω with Lipschitz constant L. It is follows from (3.6) that

$$\left\| 4r^4 f(x) - 4f(rx) \right\| \le \alpha(x, 0, 0) \tag{4.5}$$

for all $x \in X$. It is follows from (4.5) that

$$\left\| r^4 f(x) - f(rx) \right\| \le \frac{\alpha(x,0,0)}{4} \tag{4.6}$$

for all $x \in X$. Using the definition of $\gamma(x)$ in the above equation and for $i = 0$, we get

$$\left\| f(x) - \frac{f(rx)}{r^4} \right\| \le \frac{1}{r^4} \gamma(x) \Rightarrow \left\| f(x) - Tf(x) \right\| \le L\gamma(x)$$

for all $x \in X$. Hence, we obtain

$$d(Tf, f) \le L = L^{1-i} \tag{4.7}$$

for all $x \in X$. Replacing x by $\frac{x}{r}$ in (4.6), we have

$$\left\| r^4 f\left(\frac{x}{r} \right) - f(x) \right\| \le \frac{1}{4} \alpha\left(\frac{x}{r}, 0, 0 \right) \tag{4.8}$$

for all $x \in X$. Using the definition of $\gamma(x)$ in the above equation for $i = 1$, we have

$$\left\| r^4 f\left(\frac{x}{r} \right) - f(x) \right\| \le \gamma(x) \Rightarrow \left\| Tf(x) - f(x) \right\| \le \gamma(x)$$

for all $x \in X$. Hence, we get

$$d(f, Tf) \le r^4 = L^{1-i} \tag{4.9}$$

for all $x \in X$. From (4.7) and (4.9), we can conclude

$$d(f, Tf) \le L^{1-i} < \infty \tag{4.10}$$

for all $x \in X$. Now from the fixed point alternative in both cases, it follows that there exists a fixed point Q of T in Ω such that

$$Q(x) = \lim_{k \to \infty} \frac{f\left(\eta_i^k x \right)}{\eta_i^{4k}} \tag{4.11}$$

DOI : 10.18231/2454-9150.2018.0899

for all $x \in X$. In order to prove $Q: X \to Y$ satisfies the functional equation (1.3), the proof is similar to that of Theorem 3.1. Since Q is unique fixed point of T in the set $\Delta = \{f \in \Omega / d(f,Q) < \infty\}$. Therefore Q is an unique function such that

$$d(f,Q) \le \frac{1}{1-L} d(f,Tf) \Rightarrow d(f,Q) \le \frac{L^{1-i}}{1-L}$$

$$i.e., \|f(x) - Q(x)\| \le \frac{L^{1-i}}{1-L} \gamma(x)$$

for all $x \in X$. This completes the proof of the Theorem.

The following corollary is a immediate consequence of Theorem 4.1concerning the stability of (1.3).

Corollary 4.4 Let ε and s be a non-negative real numbers. If a function $P: X \to Y$ satisfies the inequality

$$\|F(x_1,x_2,x_3)\| \le \begin{cases} \varepsilon, \\ \varepsilon\left\{\sum_{i=1}^{3}\|x_i\|^s\right\}, \\ \varepsilon\left\{\prod_{i=1}^{3}\|x_i\|^s + \sum_{i=1}^{3}\|x_i\|^{3s}\right\}, \end{cases} \quad (4.12)$$

. Then there exists an unique quartic function such that

$$\|f(x) - Q(x)\| \le \begin{cases} \frac{\varepsilon}{4|r^4 - 1|} \\ \frac{\varepsilon\|x\|^s}{4|r^4 - r^s|} & ; s \ne 4 \\ \frac{\varepsilon\|x\|^{3s}}{4|r^4 - r^{3s}|} & ; s \ne \frac{4}{3} \end{cases} \quad (4.13)$$

for all $x \in X$.

Proof. Setting

$$\alpha(x_1,x_2,x_3) \le \begin{cases} \varepsilon, \\ \varepsilon\left\{\sum_{i=1}^{3}\|x_i\|^s\right\}, \\ \varepsilon\left\{\prod_{i=1}^{3}\|x_i\|^s + \sum_{i=1}^{3}\|x_i\|^{3s}\right\}, \end{cases}$$

for all $x_1, x_2, x_3 \in X$. Now

$$\frac{\alpha(\eta_i^k x_1, \eta_i^k x_2, \eta_i^k x_3)}{\eta_i^{4k}} = \begin{cases} \frac{\varepsilon}{\eta_i^{4k}}, \\ \frac{\varepsilon}{\eta_i^{4k}}\left\{\sum_{i=1}^{3}\|\eta_i x_i\|^s\right\}, \\ \frac{\varepsilon}{\eta_i^{4k}}\left\{\prod_{i=1}^{3}\|\eta_i x_i\|^s + \sum_{i=1}^{3}\|\eta_i x_i\|^{3s}\right\}, \end{cases}$$

$$= \begin{cases} \to 0 \ as \ k \to \infty \\ \to 0 \ as \ k \to \infty \\ \to 0 \ as \ k \to \infty \end{cases}$$

i.e., (4.1) is holds. Since, we have

$$\gamma(x) = \frac{1}{4}\alpha\left(\frac{x}{r}, 0, 0\right)$$

then

$$\gamma(x) = \frac{1}{4}\alpha\left(\frac{x}{r}, 0, 0\right) = \begin{cases} \frac{\varepsilon}{4} \\ \frac{\varepsilon\|x\|^s}{4r^s} \\ \frac{\varepsilon\|x\|^{3s}}{4r^s} \end{cases}$$

Also,

$$\frac{1}{\eta_i^4}\gamma(\eta_i x) = \begin{cases} \frac{1}{\eta_i^4}\frac{\varepsilon}{4} \\ \frac{1}{\eta_i^4}\frac{\varepsilon\|x\|^s \eta_i^s}{4r^s} \\ \frac{1}{\eta_i^4}\frac{\varepsilon\|x\|^{3s}\eta_i^{3s}}{4r^{3s}} \end{cases} = \begin{cases} \eta_i^{-4}\gamma(x) \\ \eta_i^{s-4}\gamma(x) \\ \eta_i^{3s-4}\gamma(x) \end{cases}$$

for all $x \in X$. Hence the inequality (4.3) holds for following cases:

$L = r^{-4}$ if $i = 0$ and $L = r^4$ if $i = 1$

$L = r^{s-4}$ for $s < 4$ if $i = 0$ and $L = r^{4-s}$ for $s > 4$ if $i = 1$

$L = r^{3s-4}$ for $s < \frac{4}{3}$ if $i = 0$ and $L = r^{4-3s}$ for $s > \frac{4}{3}$ if $i = 1$

Now from (4.4), we prove the following cases.

Case 1. $L = r^{-4}$ if $i = 0$

$$\|f(x) - Q(x)\| \le \frac{L^{1-i}}{1-L}\gamma(x) = \frac{r^{-4}}{1-r^{-4}}\frac{\varepsilon}{4} = \frac{\varepsilon}{4(r^4 - 1)}$$

Case 2. $L = r^4$ if $i = 1$

$$\|f(x) - Q(x)\| \le \frac{L^{1-i}}{1-L}\gamma(x) = \frac{1}{1-r^4}\frac{\varepsilon}{4} = \frac{\varepsilon}{4(1-r^4)}$$

Case 3. $L = r^{s-4}$ for $s < 4$ if $i = 0$

$$\|f(x) - Q(x)\| \le \frac{L^{1-i}}{1-L}\gamma(x) = \frac{r^{s-4}}{1-r^{s-4}}\frac{\varepsilon\|x\|^s}{4r^s} = \frac{\varepsilon\|x\|^s}{4(r^4 - r^s)}$$

Case 4. $L = r^{4-s}$ for $s > 4$ if $i = 1$

 DOI : 10.18231/2454-9150.2018.0899

$$\left\|f(x)-Q(x)\right\| \le \frac{L^{1-i}}{1-L}\gamma(x) = \frac{1}{1-r^{4-s}}\frac{\varepsilon\|x\|^s}{4r^s} = \frac{\varepsilon\|x\|^s}{4\left(r^s-r^4\right)}$$

Case 5. $L = r^{3s-4}$ for $s < \frac{3}{4}$ if $i = 0$

$$\left\|f(x)-Q(x)\right\| \le \frac{L^{1-i}}{1-L}\gamma(x) = \frac{r^{3s-4}}{1-r^{3s-4}}\frac{\varepsilon\|x\|^{3s}}{4r^{3s}} = \frac{\varepsilon\|x\|^{3s}}{4\left(r^4-r^{3s}\right)}$$

Case 6. $L = r^{4-3s}$ for $s > \frac{3}{4}$ if $i = 1$

$$\left\|f(x)-Q(x)\right\| \le \frac{L^{1-i}}{1-L}\gamma(x) = \frac{1}{1-r^{4-3s}}\frac{\varepsilon\|x\|^{3s}}{4r^{3s}} = \frac{\varepsilon\|x\|^{3s}}{4\left(4^{3s}-r^4\right)}$$

Hence the proof is complete.

Banach Algebra Stability Results for (1.3)

For sections 5 and 6, let us consider X and Y to a normed algebra and a Banach algebra, respectively. For notational handiness, we define a function $P: X \to Y$ by

$$\begin{aligned}P(x_1,x_2,x_3) &= f\left(rx_1+r^2x_2+r^3x_3\right)+f\left(-rx_1+r^2x_2+r^3x_3\right)\\&+f\left(rx_1-r^2x_2+r^3x_3\right)+f\left(rx_1+r^2x_2-r^3x_3\right)-2\Big[f\left(rx_1+r^2x_2\right)\\&+f\left(r^2x_2+r^3x_3\right)+f\left(rx_1+r^3x_3\right)+f\left(rx_1-r^2x_2\right)+f\left(r^2x_2-r^3x_3\right)\\&+f\left(rx_1-r^3x_3\right)\Big]+2[r^4\left(f(x_1)+f(-x_1)\right)]+r^8\left(f(x_2)+f(-x_2)\right)\\&+r^{12}\left(f(x_3)+f(-x_3)\right)\Big]+\Big[r\left(f(x_1)-f(-x_1)\right)+r^2\left(f(x_2)-f(-x_2)\right)\\&+r^3\left(f(x_3)-f(-x_3)\right)\Big]\end{aligned}$$

for all $x_1, x_2, x_3 \in X$.

V. Stability of 3-dimensional Functional Equation (1.3)-Direct Method

In this section, the authors investigate the generalized Ulam-Hyers stability of the 3-dimensional functional equation (1.3) in Banach algebra with the help of direct method.

Definition 5.1 Let X be Banach Algebra. A mapping $f: X \to X$ is said to be additive derivation if the additive function f satisfies,

$$f(x_1x_2) = f(x_1)x_2 + x_1 f(x_2)$$

for all $x_1, x_2 \in X$. Also the additive derivation for three variables satisfies

$$f(x_1x_2x_3) = f(x_1)x_2x_3 + x_1 f(x_2)x_3 + x_1x_2 f(x_3)$$

for all $x_1, x_2, x_3 \in X$.

Proposition 5.2 Let $j = \pm 1$. Let $P: X \to Y$ be a mapping for which there exists function $\alpha, \beta : X^3 \to [0,\infty)$ with the condition

$$\sum_{k=0}^{\infty}\frac{\alpha\left(r^{kj}x_1, r^{kj}x_2, r^{kj}x_3\right)}{r^{kj}} \text{ converges in } \mathbb{R} \text{ and}$$

$$\lim_{k\to\infty}\frac{\alpha\left(r^{kj}x_1, r^{kj}x_2, r^{kj}x_3\right)}{r^{kj}} = 0$$

and also

$$\sum_{k=0}^{\infty}\frac{\beta\left(r^{kj}x_1, r^{kj}x_2, r^{kj}x_3\right)}{r^{3kj}} \text{ converges in } \mathbb{R} \text{ and}$$

$$\lim_{k\to\infty}\frac{\beta\left(n^{kj}x_1, n^{kj}x_2, n^{kj}x_3\right)}{n^{3kj}} = 0$$

such that the functional inequalities

$$\left\|P(x_1,x_2,x_3)\right\| \le \alpha(x_1,x_2,x_3)$$

and

$$\left\|P(x_1x_2x_3) - P(x_1)x_2x_3 - x_1P(x_2)x_3 - x_1x_2P(x_3)\right\| \le \beta(x_1,x_2,x_3)$$

for all $x_1, x_2, x_3 \in X$. Then there exists a unique additive derivation mapping $A: X \to Y$ satisfying the functional equation (1.3) and

$$\left\|f(x)-A(x)\right\| \le \frac{1}{2r}\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\alpha\left(r^{kj}x,0,0\right)}{r^{kj}}$$

for all $x \in X$. The mapping $A(x)$ is defined by

$$A(x) = \lim_{k\to\infty}\frac{f\left(r^{kj}x\right)}{r^{kj}}$$

for all $x \in X$.

Corollary 5.3 Let $P: X \to Y$ be a mapping and there exists a real numbers ε and s such that

$$\left\|P(x_1,x_2,x_3)\right\| \le \begin{cases}\varepsilon, \\ \varepsilon\left\{\sum_{i=1}^{3}\|x_i\|^s\right\}, & s \ne 1\\ \varepsilon\left\{\prod_{i=1}^{3}\|x_i\|^s + \sum_{i=1}^{3}\|x_i\|^{3s}\right\}, & s \ne \frac{1}{3}\end{cases}$$

and

$$\left\|P(x_1x_2x_3) - P(x_1)x_2x_3 - x_1P(x_2)x_3 - x_1x_2P(x_3)\right\|$$

$$\le \begin{cases}\varepsilon, \\ \varepsilon\left\{\sum_{i=1}^{3}\|x_i\|^s\right\}, & s \ne 1\\ \varepsilon\left\{\prod_{i=1}^{3}\|x_i\|^s + \sum_{i=1}^{3}\|x_i\|^{3s}\right\}, & s \ne \frac{1}{3}\end{cases}$$

DOI : 10.18231/2454-9150.2018.0899

for all $x_1, x_2, x_3 \in X$. Then there exists a unique additive derivation $A: X \to Y$ such that

$$\|f(x)-A(x)\| \le \begin{cases} \dfrac{\varepsilon}{2|r-1|} \\ \dfrac{\varepsilon\|x\|^s}{2|r-r^s|} & ; s \ne 1 \\ \dfrac{\varepsilon\|x\|^{3s}}{2|r-r^{3s}|} & ; s \ne \dfrac{1}{3} \end{cases}$$

for all $x \in X$.

Definition 5.4 Let X be Banach Algebra. A mapping $f: X \to X$ is said to be quartic derivation if the quartic function f satisfies,

$$f(x_1x_2) = f(x_1)x_2^4 + x_1^4 f(x_2)$$

for all $x_1, x_2 \in X$. Also the quartic derivation for three variables satisfies

$$f(x_1x_2x_3) = f(x_1)x_2^4x_3^4 + x_1^4 f(x_2)x_3^4 + x_1^4x_2^4 f(x_3)$$

for all $x_1, x_2, x_3 \in X$.

Proposition 5.5 Let $j = \pm 1$. Let $P: X \to Y$ be a mapping for which there exists function $\alpha, \beta: X^3 \to [0,\infty)$ with the condition

$$\sum_{k=0}^{\infty} \frac{\alpha\left(r^{kj}x_1, r^{kj}x_2, r^{kj}x_3\right)}{r^{4kj}} \text{ converges in } \mathbb{R} \text{ and}$$

$$\lim_{k\to\infty} \frac{\alpha\left(r^{kj}x_1, r^{kj}x_2, r^{kj}x_3\right)}{r^{4kj}} = 0 \quad \text{and also}$$

$$\sum_{k=0}^{\infty} \frac{\beta\left(r^{kj}x_1, r^{kj}x_2, r^{kj}x_3\right)}{r^{12kj}} \text{ converges in } \mathbb{R} \text{ and}$$

$$\lim_{k\to\infty} \frac{\beta\left(n^{kj}x_1, n^{kj}x_2, n^{kj}x_3\right)}{n^{12kj}} = 0$$

such that the functional inequalities

$$\|P(x_1, x_2, x_3)\| \le \alpha(x_1, x_2, x_3)$$

and

$$\left\|P(x_1x_2x_3) - P(x_1)x_2^4x_3^4 - x_1^4P(x_2)x_3^4 - x_1^4x_2^4P(x_3)\right\| \le \beta(x_1, x_2, x_3)$$

for all $x_1, x_2, x_3 \in X$. Then there exists a unique quartic derivation mapping $Q: X \to Y$ satisfying the functional equation (1.3) and

$$\|f(x)-Q(x)\| \le \frac{1}{4r^4} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\alpha\left(r^{kj}x, 0, 0\right)}{r^{4kj}}$$

for all $x \in X$. The mapping $Q(x)$ is defined by

$$Q(x) = \lim_{k\to\infty} \frac{f\left(r^{kj}x\right)}{r^{4kj}}$$

for all $x \in X$.

Corollary 5.6 Let $P: X \to Y$ be a mapping and there exists a real numbers ε and s such that

$$\|P(x_1, x_2, x_3)\| \le \begin{cases} \varepsilon, \\ \varepsilon\left\{\sum_{i=1}^{3}\|x_i\|^s\right\}, & s \ne 4 \\ \varepsilon\left\{\prod_{i=1}^{3}\|x_i\|^s + \sum_{i=1}^{3}\|x_i\|^{3s}\right\}, & s \ne \dfrac{4}{3} \end{cases}$$

and

$$\left\|P(x_1x_2x_3) - P(x_1)x_2^4x_3^4 - x_1^4P(x_2)x_3^4 - x_1^4x_2^4P(x_3)\right\|$$

$$\le \begin{cases} \varepsilon, \\ \varepsilon\left\{\sum_{i=1}^{3}\|x_i\|^s\right\}, & s \ne 4 \\ \varepsilon\left\{\prod_{i=1}^{3}\|x_i\|^s + \sum_{i=1}^{3}\|x_i\|^{3s}\right\}, & s \ne \dfrac{4}{3} \end{cases}$$

for all $x_1, x_2, x_3 \in X$. Then there exists a unique quartic derivation $Q: X \to Y$ such that

$$\|f(x)-Q(x)\| \le \begin{cases} \dfrac{\varepsilon}{4|r^4-1|} \\ \dfrac{\varepsilon\|x\|^s}{4|r^4-r^s|} & ; s \ne 4 \\ \dfrac{\varepsilon\|x\|^{3s}}{4|r^4-r^{3s}|} & ; s \ne \dfrac{4}{3} \end{cases}$$

for all $x \in X$.

VI. STABILITY OF 3-DIMENSIONAL FUNCTIONAL EQUATION (1.3)-FIXED POINT METHOD

In this section, the authors the generalized Ulam-Hyers stability of the 3-dimensional functional equation (1.3) in Banach algebra with the help of fixed point method.

Proposition 6.1 Let $j = \pm 1$. Let $P: X \to Y$ be a mapping for which there exists functions $\alpha, \beta: X^3 \to [0,\infty)$ with the conditions

 DOI : 10.18231/2454-9150.2018.0899

$\sum_{k=0}^{\infty} \frac{\alpha\left(\eta_i^{kj} x_1, \eta_i^{kj} x_2, \eta_i^{kj} x_3\right)}{\eta_i^{kj}}$ converges in and

$$\lim_{k\to\infty} \frac{\alpha\left(\eta_i^{kj} x_1, \eta_i^{kj} x_2, \eta_i^{kj} x_3\right)}{\eta_i^{kj}} = 0$$

and also

$\sum_{k=0}^{\infty} \frac{\beta\left(\eta_i^{kj} x_1, \eta_i^{kj} x_2, \eta_i^{kj} x_3\right)}{\eta_i^{3kj}}$ converges in and

$$\lim_{k\to\infty} \frac{\beta\left(\eta_i^{kj} x_1, \eta_i^{kj} x_2, \eta_i^{kj} x_3\right)}{\eta_i^{3kj}} = 0$$

where $\eta_i = \begin{cases} r, & i = 0; \\ \frac{1}{r} & i = 1; \end{cases}$ satisfying the functional inequalities

$$\left\| P(x_1, x_2, x_3) \right\| \le \alpha(x_1, x_2, x_3)$$

and

$$\left\| P(x_1 x_2 x_3) - P(x_1) x_2 x_3 - x_1 P(x_2) x_3 - x_1 x_2 P(x_3) \right\| \le \beta\left(x_1, x_2, x_3\right)$$

for all $x_1, x_2, x_3 \in X$. Then there exists $L = L(i) < 1$ such that the function

$$x \to \beta(x) = \frac{1}{2} \alpha\left(\frac{x}{r}, 0, 0\right)$$

has the property

$$\frac{1}{\eta_i} \beta\left(\eta_i x\right) = L \beta(x)$$

for all $x \in X$. Then there exists a unique additive derivation mapping $A: X \to Y$ satisfying the functional equation (1.3) and

$$\left\| f(x) - A(x) \right\| \le \frac{L^{1-i}}{1-L} \beta(x)$$

for all $x \in X$.

Corollary 6.2 Let $P: X \to Y$ be a mapping and there exists a real numbers ε and s such that,

$$\left\| P\left(x_1, x_2, x_3\right) \right\| \le \begin{cases} \varepsilon, & \\ \varepsilon \left\{ \sum_{i=1}^{3} \left\| x_i \right\|^s \right\}, & s \ne 1 \\ \varepsilon \left\{ \prod_{i=1}^{3} \left\| x_i \right\|^s + \sum_{i=1}^{3} \left\| x_i \right\|^{3s} \right\}, & s \ne \frac{1}{3} \end{cases}$$

and

$$\left\| P(x_1 x_2 x_3) - P(x_1) x_2 x_3 - x_1 P(x_2) x_3 - x_1 x_2 P(x_3) \right\|$$

$$\le \begin{cases} \varepsilon, & \\ \varepsilon \left\{ \sum_{i=1}^{3} \left\| x_i \right\|^s \right\}, & s \ne 1 \\ \varepsilon \left\{ \prod_{i=1}^{3} \left\| x_i \right\|^s + \sum_{i=1}^{3} \left\| x_i \right\|^{3s} \right\}, & s \ne \frac{1}{3} \end{cases}$$

for all $x_1, x_2, x_3 \in X$. Then there exists a unique additive derivation $A: X \to Y$ such that

$$\left\| f\left(x\right) - A\left(x\right) \right\| \le \begin{cases} \frac{\varepsilon}{2\left| r-1 \right|} & \\ \frac{\varepsilon \left\| x \right\|^s}{2\left| r - r^s \right|} & ; s \ne 1 \\ \frac{\varepsilon \left\| x \right\|^{3s}}{2\left| r - r^{3s} \right|} & ; s \ne \frac{1}{3} \end{cases}$$

for all $x \in X$.

Proposition 6.3 Let $j = \pm 1$. Let $P: X \to Y$ be a mapping for which there exists functions $\alpha, \beta: X^3 \to [0, \infty)$ with the conditions $\sum_{k=0}^{\infty} \frac{\alpha\left(\eta_i^{kj} x_1, \eta_i^{kj} x_2, \eta_i^{kj} x_3\right)}{\eta_i^{4kj}}$ converges in and $\lim_{k\to\infty} \frac{\alpha\left(\eta_i^{kj} x_1, \eta_i^{kj} x_2, \eta_i^{kj} x_3\right)}{\eta_i^{4kj}} = 0$ and also

$\sum_{k=0}^{\infty} \frac{\beta\left(\eta_i^{kj} x_1, \eta_i^{kj} x_2, \eta_i^{kj} x_3\right)}{\eta_i^{12kj}}$ converges in and

$$\lim_{k\to\infty} \frac{\beta\left(\eta_i^{kj} x_1, \eta_i^{kj} x_2, \eta_i^{kj} x_3\right)}{\eta_i^{12kj}} = 0$$

where $\eta_i = \begin{cases} r, & i = 0; \\ \frac{1}{r} & i = 1; \end{cases}$ satisfying the functional inequalities

$$\left\| P(x_1, x_2, x_3) \right\| \le \alpha(x_1, x_2, x_3)$$

and

$$\left\| P(x_1 x_2 x_3) - P(x_1) x_2^4 x_3^4 - x_1^4 P(x_2) x_3^4 - x_1^4 x_2^4 P(x_3) \right\| \le \beta\left(x_1, x_2, x_3\right)$$

for all $x_1, x_2, x_3 \in X$. Then there exists $L = L(i) < 1$ such that the function

$$x \to \beta(x) = \frac{1}{4} \alpha\left(\frac{x}{r}, 0, 0\right)$$

has the property

 DOI : 10.18231/2454-9150.2018.0899

$$\frac{1}{\eta_i^4}\beta\left(\eta_i x\right) = L\beta(x)$$

for all $x \in X$. Then there exists a unique derivation mapping $Q: X \to Y$ satisfying the functional equation (1.3) and

$$\left\| f(x) - Q(x) \right\| \le \frac{L^{1-i}}{1-L}\beta(x)$$

for all $x \in X$.

Corollary 6.4 Let $P: X \to Y$ be a mapping and there exists a real numbers ε and s such that,

$$\left\|P\left(x_1, x_2, x_3\right)\right\| \le \begin{cases} \varepsilon, & \\ \varepsilon\left\{\sum_{i=1}^{3}\left\|x_i\right\|^s\right\}, & s \ne 4 \\ \varepsilon\left\{\prod_{i=1}^{3}\left\|x_i\right\|^s + \sum_{i=1}^{3}\left\|x_i\right\|^{3s}\right\}, & s \ne \frac{4}{3} \end{cases}$$

and

$$\left\|P(x_1x_2x_3) - P(x_1)x_2^4x_3^4 - x_1^4P(x_2)x_3^4 - x_1^4x_2^4P(x_3)\right\|$$

$$\le \begin{cases} \varepsilon, & \\ \varepsilon\left\{\sum_{i=1}^{3}\left\|x_i\right\|^s\right\}, & s \ne 4 \\ \varepsilon\left\{\prod_{i=1}^{3}\left\|x_i\right\|^s + \sum_{i=1}^{3}\left\|x_i\right\|^{3s}\right\}, & s \ne \frac{4}{3} \end{cases}$$

for all $x_1, x_2, x_3 \in X$. Then there exists a unique derivation $Q: X \to Y$ such that

$$\left\|f\left(x\right) - Q\left(x\right)\right\| \le \begin{cases} \dfrac{\varepsilon}{4\left|r^4 - 1\right|} & \\ \dfrac{\varepsilon\left\|x\right\|^s}{4\left|r^4 - r^s\right|} & ; s \ne 4 \\ \dfrac{\varepsilon\left\|x\right\|^{3s}}{4\left|r^4 - r^{3s}\right|} & ; s \ne \dfrac{4}{3} \end{cases}$$

for all $x \in X$.

REFERENCES

[1] G. Balasubramaniyan, V. Govindan and C. Muthamilarasi, General Solution and Stability of Quadratic Functional Equation, Int. J. Math. And Appl., Vol. 5, no. 2A,13-26, 2017.

[2] V. Govindan, Stability Solution of Quadratic Functional Equation in Banach Space using Direct Method, Inter. J. Scientific Eng. Research, Vol. 1, no.1, 01-06 , 2018.

[3] V. Govindan and K. Tamilvanan, Stability of Functional Equation in Banach Space: Using Two Different Methods, Int. J. Math. And Appl., Vol. 6, no. 1-C, 527-536, 2018.

[4] D. H. Hyers, On the stability of the linear functional equation, Proc. Natl. Acad. Sci., U.S.A., Vol. 27 , 222-224, 1941.

[5] H. Khodaei and Th.M. Rassias, Approximately Generalized Additive Functions in Several Variables, Int. J. Nonlinear Anal. Appl., Vol. 1, no. 1, 22–41, 2010.

[6] S. Murthy, V. Govindhan and M. Sree Shanmuga Velan, Solution and stability of Two Types of n-Dimensional Quartic Functional Equation in Generalized 2-Normed Spaces, Int. J. Pure and Appl. Math., Vol. 111, no.2, 249-272, 2016.

[7] S. Murthy, V. Govindhan and M. Sree ShanmugaVelan, Generalized U – H Stability of New n – type of Additive Quartic Functional Equation in Non – Archimedean, Int. J. Math. And Appl., Vol. 5, no. 2A, 1-11, 2017.

[8] C.G. Park and J. M. Rassias, Hyers-Ulam stability of an Euler-Lagrange type Additive Mapping, Int. J. Appl. Math. Stat., Vol. 7(Fe07), 112–125, 2007.

[9] C. Park and T. M. Rassias, Fixed Points and Generalized Hyers-Ulam Stability of Quadratic Functional Equations, J. Math. Inequal., Vol. 1, no.4, 515–528, 2007.

[10] J. M. Rassias, On Approximation of Approximately Linear Mappings by Linear Mappings, Bull. Sci. Math., Vol. 108, no. 4, 445–446, 1984.

[11] J. M. Rassias, On a New Approximation of Approximately Linear Mappings by Linear Mappings, Discuss. Math., Vol. 7, 193–196, 1985.

[12] S. M. Ulam, Problems in Modern Mathematics, Science Editions, Wiley, New York, 1964.

 DOI : 10.18231/2454-9150.2018.0899