International Journal for Research in Engineering Application & Management (IJREAM)
ISSN : 2454-9150 Vol-04, Issue-08, Nov 2018

On idempotent circulant matrices over semirings

M. Radhakrishnan, Department of Mathematics, SRM Institute of Science and Technology, Tamil Nadu, India. Email:rakrims@gmail.com

N. Elumalai, Department of Mathematics, A.V.C. College(Autonomous), Mannampandal, Tamil Nadu, India. Email: nelumalai@rediffmail.com

Abstract A semiring is an algebraic system $(S,+,\cdot)$ in which $(S,+)$ is an abelian monoid with identity element zero and $(S,\cdot)$ is another semigroup, these semigroups are related by right and left distributive laws and accordingly the 0 is the right or left, (respectively) side absorptive element. In this paper, we examines the property of regular semiring on any idempotent circulant matrix semiring. We show that for a positive integer n, $M_n(S)$ is a regular semiring, then S is regular. We discussed the strongly regular, left regular, right regular matrix semiring for idempotent circulant matrices. We also obtain certain properties of regularities over idempotent circulant matrix semirings.

Keywords* — *Semiring, left regular, right regular, strongly regular, circulant matrix, idempotent circulant matrix.

I. INTRODUCTION

The notion of semiring was first introduced by H.S. Vandiver in 1934. H.S. Vandiver introduced an algebraic system, which consists of non empty set S with two binary operations addition $+$ and multiplication $\cdot$. The system he constructed was ring like but not exactly a ring. Vandiver called this system a 'Semiring'. The study of matrices over general semirings has a long history. In 1964, Rutherford[1] gave a proof of Cayley –Hamilton theorem for a commutative semiring avoiding the use of determinants. In 1982, Reutenauer and Straubing[2] showed that $AB = I$ implies $BA = I$ over a commutative semirings. Since then, a number of works on theory of matrices over semirings were published. In 1999, J S Golan[3] described semirings and matrices over semirings in his work. One of the most interesting topics in this area is determining the invertible matrices over a specific semiring. Invertible matrices over semirings of various types have been studied. Luce[4] characterized the invertible matrices over a Boolean algebra of atleast 2 elements. He showed that they must be orthogonal matrices. Rutherford[5] showed that a square matrix over a Boolean algebra of 2 elements is invertible if and only if it is a permutation matrix. The invertible matrices over a special commutative antiring were characterized by Tan[6]. Idempotent circulant matrix was introduced by Radhakrishnan.M and N.Elumalai et al [7]. Regular rings was orignially introduced by Von Neumann in order to clarify certain aspects of operator algebras. Regular rings have also been widely studied for their own sake. See [8] for various interesting properties of regular rings. In this paper, we study certain properties of regular semiring on any idempotent circulant matrix semiring. We show that for a positive integer n, $M_n(S)$ is a regular semiring, then S is regular. But the converse need not be true for $n = 2$. We also discuss strongly regular, left(right) strongly regular semiring for idempotent circulant matrices. Throughout this paper, Let $S = \{0,1\}$ along with operations defined by $0+0 = 0\cdot 1 = 1\cdot 0 = 0\cdot 0 = 0$ and $1+0 = 0+1 = 1+1 = 1\cdot 1 = 1$. and the $M = M_n(S)$ is set of all idempotent circulant matrices $n \times n$ matrices over S.

II. PRELIMINARIES

In this section we collect all the results concerning matrix over semiring.

Definition 2.1

Let S be a non empty set with two binary operations $+$ and $\cdot$. Then the algebraic structure $(S,+,\cdot)$ is called a semiring if $\forall\, x,y,z \in S$

1. $(S,+)$ is a semigroup
2. $(S,\cdot)$ is a semigroup
3. $x\cdot(y+z) = x\cdot y + x\cdot z$ and
   $(y+z)\cdot x = y\cdot x + z\cdot x$

Definition 2.2

Let $(S,+,\cdot)$ be a semiring. Then S is called a commutative semiring iff $\forall\ x,y \in S$

1. $x+y = y+x$
2. $x\cdot y = y\cdot x$

Example 2.3 *Clearly, $(S = \{0,1\},+,\cdot)$ is a commutative semiring.*

Definition 2.4

For any given $y_0, y_1, y_2, \cdots, y_{n-1} \in \mathbb{R}^{n\times n}$, the circulant matrix is defined by $Y = (y_{i,j}) = (y_{(j-i)modn}) =$

$$\begin{bmatrix} y_0 & y_1 & y_2 & \cdots & y_{n-2} & y_{n-1} \\ y_{n-1} & y_0 & y_1 & \cdots & y_{n-3} & y_{n-2} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ y_1 & y_2 & y_3 & \cdots & y_1 & y_0 \end{bmatrix}$$

Remark 2.5 *[9] Let X and Y be $n \times n$ circulant matrices. Then the product XY is also a circulant matrix.*

 DOI : 10.18231/2454-9150.2018.1118

ISSN : 2454-9150 Vol-04, Issue-08, Nov 2018

Remark 2.6 *[9] Let X and Y be* $n \times n$ *circulant matrices. Then XY=YX.*

Definition 2.7

Let $(M,+,\cdot)$ *is a commutative semiring, where M with the operations are defined by*

1. $[x_{ij}] + [y_{ij}] = [x_{ij} + y_{ij}]$
2. $[x_{ij}] \cdot [y_{ij}] = \sum_{k=1}^{n} [x_{ik} y_{kj}]$

Definition 2.8

If $\forall\ A \in M$, *there exists* $B \in M$ *such that* $ABA = A$, *then commutative semi ring* $(M,+,\cdot)$ *is called as a regular semiring.*

Example 2.9

Clearly, the commutative semiring,

$$\left(M_2(S) = \left\{O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, X = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right\}, +, \cdot\right)$$

is a regular semi ring.

III. MAIN RESULTS

Theorem 3.1 Let $(S,+,\cdot)$ be an additively commutative semiring with zero and n a positive integer. S is a regular semiring if M is a regular semiring.

Proof. Let M is regular semiring.

Then by definition, $\forall\ X \in M, \exists Y \in M$ such that $XYX = X$. Then

$$X = \begin{bmatrix} x & 0 & 0 & \cdots & 0 \\ 0 & x & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & x \end{bmatrix}$$

For this X, there exist a

$$Y = \begin{bmatrix} y_{11} & y_{12} & y_{13} & \cdots & y_{1n} \\ y_{21} & y_{22} & y_{23} & \cdots & y_{2n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ y_{n1} & y_{n2} & y_{n3} & \cdots & y_{nn} \end{bmatrix}$$

such that $XYX = X$

$$XYX = \begin{bmatrix} xy_{11}a & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$\therefore XYX = X \Rightarrow xy_{11}x = x$$

Remark 3.2 The converse of Proposition need not *be true for* $n = 2$, *as the following example shows. Let* $S = \{0,1\}$ *and* $(S,\oplus,\otimes)$, *the semiring with operations defined by* $x \oplus y = max\{x,y\}$ *and* $x \otimes y = min\{x,y\}$ *for all* $x, y \in S$. *Then* $(S,\oplus,\otimes)$ *is regular. Suppose that* $M_2(S)$ *is a regular semiring. Let* $X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in M_2(S)$

Then $X = XYX$ for some $Y \in M_2(S)$. Then by the definition if $\oplus$ and $\otimes$,

$$XY = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_{11} & y_{12} \\ y_{12} & y_{11} \end{bmatrix}$$

$$= \begin{bmatrix} 1 \otimes y_{11} & 1 \otimes y_{12} \\ 1 \otimes y_{12} & 1 \otimes y_{11} \end{bmatrix}$$

$$\therefore XYX = \begin{bmatrix} ((1 \otimes y_{11}) \otimes 1) & ((y_{12} \otimes 1) \otimes 1) \\ ((y_{12} \otimes 1) \otimes 1) & ((1 \otimes y_{11}) \otimes 1) \end{bmatrix}$$

Since $X = XYX$, it follows that

$$(1 \otimes y_{11}) \otimes 1 = 1 \quad (1)$$

$$(y_{12} \otimes 1) \otimes 1 = 0 \quad (2)$$

From 2, $y_{12} = 0$

From 1, $(1 \otimes y_{11}) \otimes 1 = 1$

But $(1 \otimes y_{11}) \leq 1$, we have a contradiction.

Definition 3.3 *[10] If* $\forall\ X \in M$, *there exists* $Y \in M$ *with* $X = X^2Y$, *then commutative semi ring* $(M,+,\cdot)$ *is called as a strongly regular semiring.*

Theorem 3.4 *Let* $(S,+,\cdot)$ *be an additively commutative semiring with zero and n a positive integer. S is a strongly regular if M is stongly regular.*

Proof

Let M is strongly regular.

Then by definition, $\forall\ X \in M, \exists Y \in M$ such that $X = X^2Y$. Then

$$X = \begin{bmatrix} x & 0 & 0 & \cdots & 0 \\ 0 & x & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & x \end{bmatrix}$$

For this X, there exist a

$$Y = \begin{bmatrix} y_{11} & 0 & 0 & \cdots & 0 \\ 0 & y_{11} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & y_{11} \end{bmatrix}$$

such that $X^2Y = X$

$$X^2Y = \begin{bmatrix} x^2y_{11} & 0 & 0 & \cdots & 0 \\ 0 & x^2y_{11} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & x^2y_{11} \end{bmatrix}$$

$$\therefore X^2Y = X \Rightarrow x^2y_{11} = x$$

$\therefore S$ is strongly regular.

Definition 3.5 $(M,+,\cdot)$ *is left(right) regular if* $\forall\ X \in M$, *there exists* $Y \in M$ *such that* $X = YX^2 = XYX(X = X^2Y = XYX)$.

Definition 3.6 $(M,+,\cdot)$ *is left(right) strongly regular if* $\forall\ X \in M$, *there exists* $A \in M$ *with* $X = AX^2(X = X^2A)$.

 DOI : 10.18231/2454-9150.2018.1118

ISSN : 2454-9150 Vol-04, Issue-08, Nov 2018

Remark 3.7 *A left(right) regular semiring is both regular and left(right) strongly regular semiring.*

Theorem 3.8 *Let* $(S, +, \cdot)$ *be an additively commutative semiring with zero and n a positive integer.* S *is a left(right) strongly regular if* M *is left(right) strongly regular.*

Theorem 3.9 *A left(right) regular semiring* M *is right(left) regular.*

Conclusion

In this work, a new type of matrices over semiring considered and establish the important properties. The study concludes that a semiring S is a left(right) regular if M is left(right) regular.

References

[1] Rutherford D E, "The Cayley-Hamilton theorem for semiring." *Proc. Roy. Soc. Edinburgh. Sec. A*, vol. 66, pp 211--215, 1964.

[2] C. Reutenauer and H. Straubing," Inversion of matrices over a commutative semirings," *J Algebra*, vol. 88, pp. 350--360, 1984.

[3] J. S. Golan, "Semirings and their applications," *Proc. Glasgow Math. Assoc Kluwer Academic Publishers*, 1999.

[4] R. D. Luce, "A note on Boolean matrix theory," *Proc. Am. Math. Soc.*, vol. 3, pp 382--388, 1952.

[5] D. E. Rutherford, "Inverses of Boolean matrices," *Proc. Glasgow Math. Assoc.*, vol. 6, pp. 49--53, 1993.

[6] Yi-Jia Tan, "On invertible matrices over commutative semirings," *Linearand Multilinear Algebra*, vol. 61:6 710—724, 2013.

[7] M.Radhakrishnan and N.Elumalai et al, "Idempotent circulant matrices," *Jounral of physics:Conference series*, vol. 1000 pp. 1–3, 2018.

[8] K. R. Goodearl, "Von Neumann Regular Rings," *Pitman,,London*, 1979.

[9] Cleghorn, C.S., 'Application of generalized inverses and circulant matrices to iterated functions on integers'. Department of Mathematics, Angelo State University, San Angelo, TX, USA, 2001.

[10] Gordon Mason, "Strongly Regular Near-rings," *Proceedings of the Edinburgh Mathematical Society*, vol. 23, pp 27-35, 1980.

 DOI : 10.18231/2454-9150.2018.1118