International Journal for Research in Engineering Application & Management (IJREAM)
ISSN : 2454-9150 Vol-04, Issue-08, Nov 2018

A Study on finite dimensional probabilistic normed spaces

***A.,Thenmozhi, Research scholar, Department of Mathematics, Reg no.10283, Manonmaniam Sundaranar University, Tiunelveli, India. thenmozhi090@gmail.com**

Dr. A. Singadurai, Department of Mathematics, T.D.M.N.S. College, T. Kallikulam, India Singadurai_59@yahoo.co.in

Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627012, Tamilnadu, India.

Abstract - The purpose of this paper is to study some concepts in probabilistic normed space. Also as in the usual normed space, we establish that every finite dimensional probabilistic normed linear space is a complete space. Also we establish a connection between a compact set and finite dimensional probabilistic normed space.

Keywords: Probabilistic norm, t-norm, continuous triangle mapping.

I. Introduction

In 1942 Menger[10] introduced the notion of probabilistic metric space. The idea of Menger was to use distribution function instead of non-negative real numbers as values of the metric. The concept of probabilistic normed spaces was introduced by A.N. $\tilde{S}$erstnev[12] in 1963. In 1993 Alsina, Schweizer and Sklar gave a new definition of probabilistic normed spaces which includes $\tilde{S}$erstnevś as a special case. In this paper we are interested in some properties of a finite dimensional probabilistic normed spaces. Also we establish some important results involving completeness and compactness of finite dimensional probabilistic normed spaces.

II. Prelimnaries

Definition 1.1[8]. The space of all distance distribution functions (d.d.f) is defined by $\Delta^+ = \{F:\mathbb{R}\cup\{-\infty,\infty\} \to [0,1]$ / F is left – continuous, non decreasing and $F(0) = 0\}$.Consider $D^+ = \{F\in\Delta^+ : \lim_{t\to\infty} F(t) = 1\}$.

By setting F≤G whenever $F(t) \le G(t)$ for all $t \in\mathbb{R}^+$, one introduces a natural ordering in D^+. Define the step function H(t) as

$$H(t) = \begin{cases} 0 \text{ if } t \le 0 \\ 1 \text{ if } t > 0. \end{cases}$$

It is clear that $H\in D^+$.

Definition 1.2[10]. A t-norm is a function $T:[0,1]\times[0,1]\to[0,1]$ which is associative, commutative, non decreasing in each place and such that $T(a,1) = a$, for all $a\in[0,1]$.

Definition 1.3[4]. A continuous triangle mapping is $\tau:\Delta^+ x\Delta^+\to\Delta^+$ which is associative, commutative, non decreasing continuous and for which H is the identity, that is $\tau(H,F)=F$ for every $F\in D^+$.

Definition1.4[4]. Let X be a vector space over $\mathbb{R}$ or $\mathbb{C}$ and τ be a continuous triangle mapping. A mapping $\mathcal{F}:X\to D^+$ satisfying the conditions

(i) $F_x=H$ if and only if $x=\theta$, the zero element in X.

(ii) $F_{\alpha x}(t)= F_x\left[\frac{t}{|\alpha|}\right]$

(iii) $F_{x+y} \ge\tau(F_x, F_y)$

is called probabilistic norm. The triple $(X, \mathcal{F},\tau)$ is called probabilistic normed space. In the above definition the value of $\mathcal{F}$ at x is denoted by F_x.

Note 1.5[9]. Suppose the condition (iii), of the above definition 1.4 is replaced by the condition $F_{x+y}(t_1 + t_2) \ge T\,(F_x(t_1), F_y(t_2))$ for all $t_1, t_2 >0$ where T is a t-norm. Then $(X, \mathcal{F},T)$ is called Random normed space and $\mathcal{F}$ is called Random norm.

 DOI : 10.18231/2454-9150.2018.1120

Example 1.6[16]. Let (X, ‖.‖) be a usual normed space. Define $F_x(t)$ = H(t–‖x‖).Then (X, $\mathcal{F}$,T) is a random normed space, where T is any t – norm.

Example 1.7[16]. Suppose T is any continuous t–norm and

τ_T (F,G)(t) = $\underset{t_1+t_2\le t}{Sup}$ {T(F (t$_1$), G (t$_2$))}. Then (X,$\mathcal{F}$,τ_T) is a probabilistic normed space where $\mathcal{F}$ is as given in example 1.6.

Definition 1.8[5]. Let X be a vector space over ℝ or ℂ and a mapping $\mathcal{F}$:X→D$^+$. Then the pair (X, $\mathcal{F}$) is said to be probabilistic semi normed space (PSN space) if F_x satisfies the following conditions .

(i) F_x=H if and only if x=θ

(ii) F_x=F_{-x} .

Example 1.9. Let (X, ‖.‖) be a usual normed space. Define $F_x(t)$=H(t–‖x‖). Then (X,$\mathcal{F}$,τ) is a probabilistic normed space where τ(F, G)(t)=$min_{t_1+t_2=t}$ {F(t_1), G(t_2)}.

Now, we prove the conditions of probabilistic norm.

(i) $x=\theta \Leftrightarrow$ ‖x‖= 0 ⇔ $F_x(t)$= H(t) ⇔F_x = H

Hence F_x= H if and only if $x=\theta$.

(ii) Let α be a scalar, x ∊ X. Then $F_{\alpha x}(t)$ = H (t – ‖ αx‖) = H (t–|α| ‖ x ‖).

Now, H (t–|α| ‖ x‖) = $\begin{cases} 0 \text{ if } t/|\alpha| \le \|x\| \\ 1 \text{ if } t/|\alpha| > \|x\| \end{cases}$

Also F_x (t/|α|) = H ((t/|α|)–‖x‖) = $\begin{cases} 0 \text{ if } t/|\alpha| \le \|x\| \\ 1 \text{ if } t/|\alpha| > \|x\| \end{cases}$

Hence $F_{\alpha x}$(t)= F_x(t/|α|).

(iii)Since τ(F_x, F_y)(t)= $min_{t_1+t_2=t}$ {$F_x(t_1)$, $F_y(t_2)$}, we have τ(F_x, F_y)(t)= $min_{t_1+t_2=t}$ {H(t_1–‖x‖), H(t_2–‖y‖)}. Hence τ(F_x, F_y)(t)=1 if and only if $F_x(t_1)$=1 and $F_x(t_2)$=1for all t_1 , t_2 with $t_1+t_2=t$. This means that τ(F_x, F_y)(t)=1 if and only if t_1> ‖ x‖ and t_2> ‖y‖ 1for all t_1 , t_2 with $t_1+t_2=t$. Now ‖ x+y‖≤‖ x‖+‖y‖< $t_1+t_2=t$. Hence ‖ x+y‖<t. This means that $F_{x+y}(t)$=1. Hence whenever that τ Since τ(F_x, F_y)(t)=1 we have $F_{x+y}(t)$=1 and so F_{x+y}(t)≥ τ(F_x, F_y)(t).

Definition 1.10[16]. Let (X, $\mathcal{F}$,τ) be a probabilistic normed space. Then a sequence {x_n}in X is said to converge to a point x∊ X, denoted by $x_n \to x$ if $\lim_{n\to\infty} F_{x_n-x}$ =H. That is for any t>0, 0<ε<1, there is a natural number N such that F_{x_n-x} (t)>1–ε for all n≥N.

Definition 1.11[16]. A sequence {x_n} in X is said to be a Cauchy sequence if

$F_{x_n-x_m}$ =H whenever n, m→∞. A probabilistic normed space (X, $\mathcal{F}$,τ) is complete if every Cauchy sequence converges.

III. INDUCED NORMED SPACES AND INDUCED PROBABILISTIC NORMED SPACES

Here we establish a link between usual normed space and probabistic normed space.

Theorem 2.1. Let (X,$\mathcal{F}$,τ) be a probabilistic normed space as given in example 1.9. Define‖x‖$_\alpha$=inf{t>0 : F_x(t)>1–α}. Suppose F_x(t)>0 for all t>0. Then $x=\theta$.

Proof. Now F_x(t) = H(t–‖ x‖) = $\begin{cases} 0 \text{ if } t \le \|x\| \\ 1 \text{ if } t > \|x\|. \end{cases}$

Hence F_x(t)>0 for all t>0 means F_x(t)=1 for all t>0. By definition of F_x, we have ‖ x‖<t for all t>0. This means that ‖ x‖=0 and so $x=\theta$, the zero element in X.

Theorem 2.2. Let (X,$\mathcal{F}$,τ) be a probabilistic normed space in which

F_x(t)>0 for all t>0 implies $x=\theta$ and τ(F,G) = Min{F,G}.Then (X, ‖.‖$_\alpha$) is a normed space for each α∊(0, 1) and {‖ x‖$_\alpha$: α∊(0, 1)} is a descending family of norms on X.

 DOI : 10.18231/2454-9150.2018.1120

Proof. As in theorem 2.5 of [15], (X, $\|.\|_\alpha$) is a normed space. Let $\alpha_1 < \alpha_2$. By definition, $\| x \|_{\alpha_1}$= inf{t: F_x(t)>$1-\alpha_1$} and $\| x \|_{\alpha_2}$ = inf {t: F_x(t) > $1-\alpha_2$}. Since $\alpha_1 < \alpha_2$, it is clear that {t: F_x(t)>$1-\alpha_1$}$\subset$ {t: F_x(t)>$1-\alpha_2$ }. Hence $\| x \|_{\alpha_2} \leq \| x \|_{\alpha_1}$.

Remark 2.3. The usual normed space (X, $\|.\|_\alpha$) obtained in the above theorem is called induced normed space.

Theorem 2.4[16]. Let (X,$\|.\|$)be a normed space. Let $\mathcal{F}$:X→D$^+$ as $F_x(t)=\begin{cases} \frac{t}{t+\|x\|} & \text{if } t>0 \\ 0 & \text{if } t\leq 0 \end{cases}$

Then (X, $\mathcal{F}$,τ_M) is a probabilistic normed space where M is the minimum t-norm.

Remark 2.5. The probabilistic normed space (X, $\mathcal{F}$,τ_M) obtained in the above theorem is called induced probabilistic normed space.

Theorem 2.6. Suppose X=$\mathbb{R}^2$, τ(F, G)(t)= F(t).G(t), the usual product of F(t) and G(t). For x=(x_1, x_2) in X define F_x as

$$F_x(t) = \begin{cases} \frac{t^2}{(t+|x_1|)(t+|x_2|)} & \text{if } t>0 \\ 0 & \text{if } t\leq 0 \end{cases}$$

Let $\mathcal{F}$ be the set of all such F_x. Then (X,$\mathcal{F}$,τ) is a probabilistic normed space.

Remrk 2.7. The conditions given in theorem 2.2 are essential for getting induced normed space (X, $\|.\|_\alpha$).

Theorem 2.8. Suppose X=$\mathbb{R}^2$, τ(F, G)= Min {F, G}. For x=(x_1, x_2) in X define F_x as

$$F_x(t) = \begin{cases} \frac{t^2}{(t+|x_1|)(t+|x_2|)} & \text{if } t> 0 \\ 0 & \text{if } t\leq 0. \end{cases}$$

Let $\mathcal{F}$ be the set of all such F_x. Then (X, $\mathcal{F}$, τ) is not a probabilistic normed space. However (X, $\mathcal{F}$, τ) is probabilistic semi normed space.

Proof. The conditions (i) of definition 1.4 is proved in theorem 2.6. Now take x=(1, 0) and y=(0, 1). Then x+y=(1, 1) and $F_x(t)= \frac{t^2}{(t+1)(t+0)} = \frac{t^2}{(t+1)t} = \frac{t}{(t+1)}$. Hence $F_x(1) = \frac{1}{2}$. Similarly $F_y(1) = \frac{1}{2}$. Now F_{x+y} (t)= $F_{(1,1)}(t) = \frac{t^2}{(t+1)(t+1)}$. Hence $F_{(1,1)}(1)=\frac{1}{4}$. This implies that $F_{x+y}(1)= \frac{1}{4}$<Min{ F_x(1), F_y (1)}. Hence (X, $\mathcal{F}$,τ) is not a probabilistic normed space. Now F_{-x}(t) (t) = $\frac{t^2}{(t+|x_1|)(t+|x_2|)}$ = F_x(t) and so (X, $\mathcal{F}$, τ) is probabilistic semi normed space.

IV. FINITE DIMENSIONAL PROBABILISTIC NORMED SPACES

Definition 3.1[13] . Let (X,$\mathcal{F}$,τ) be a probabilistic normed space.

i) The element {x_1, x_2,… x_n} of X is linearly dependent, if there exists k_1, k_2,… k_n not all zero such that $F_{k_1x_1+k_2x_2+\cdots+k_nx_n}$(t)=H(t), if finite set {x_1, x_2,… x_n}is not linearly dependent, it is called linearly independent.

ii) The element {x_1, x_2,… x_n} of X is called a basis if {x_1, x_2,… x_n} are linearly independent and if any element of X is a linear combination of element t {x_1, x_2,… x_n} .

The X is called a n-dimensional, if X has a basis of n elements.

Lemma 3.2. Let (X,$\mathcal{F}$,τ) be a probabilistic normed space. Let {x_1, x_2} be a linearly independent set of vectors. Then there exists c>0, δ ∈(0,1) such that for any two scalars α_1, α_2 we have $F_{\alpha_1x_1+\alpha_2x_2}$(c $|\alpha_1|$+c $|\alpha_2|$)<$1-\delta$.

Proof. Take s= $|\alpha_1|$+ $|\alpha_2|$. If s=0 then α_1=α_2=0. Hence $F_{\alpha_1x_1+\alpha_2x_2}$(c $|\alpha_1|$+c $|\alpha_2|$)= F_θ(0)= H(0) =0. This means that $F_{\alpha_1x_1+\alpha_2x_2}$(c $|\alpha_1|$+c $|\alpha_2|$)<$1-\delta$ for all δ ∈(0,1) and for all c>0. Suppose s>0. Then $|\frac{\alpha_1}{s}|$+ $|\frac{\alpha_2}{s}|$=1. Take $\beta_1=\frac{\alpha_1}{s}$ and $\beta_2=\frac{\alpha_2}{s}$, then $\sum_{i=1,2} |\beta_i|$=1. Now c $|\alpha_1|$+c $|\alpha_2|$ = c $|\beta_1 s|$+c $|\beta_2 s|$ = cs($|\beta_1|$+ $|\beta_2|$) = cs. Hence $F_{\alpha_1x_1+\alpha_2x_2}$(c $|\alpha_1|$+c $|\alpha_2|$)= $F_{s\beta_1x_1+s\beta_2x_2}$(c($|\beta_1 s|$+ $|\beta_2 s|$))= $F_{s(\beta_1x_1+\beta_2x_2)}$(cs($|\beta_1|$+ $|\beta_2|$)) =$F_{\beta_1x_1+\beta_2x_2}$(c). Hence $F_{\alpha_1x_1+\alpha_2x_2}$(c $|\alpha_1|$+c $|\alpha_2|$) = $F_{\beta_1x_1+\beta_2x_2}$ (c). Suppose the result $F_{\alpha_1x_1+\alpha_2x_2}$(c $|\alpha_1|$+c $|\alpha_2|$)<$1-\delta$ does not hold. Then for each c>0 and δ ∈(0,1) there is a set β_1, β_2 with $\sum_{i=1,2} |\beta_i|$=1 such that $F_{\beta_1x_1+\beta_2x_2}(c)\geq 1-\delta$. Take c=$\delta$=$\frac{1}{m}$, m an integer. Then corresponding to each m there are scalar ${\beta_1}^{(m)}$>${\beta_2}^{(m)}$ with $|{\beta_1}^{(m)}|$+ $|{\beta_2}^{(m)}|$ =1 such that $F_{{\beta_1}^{(m)}x_1+{\beta_2}^{(m)}x_2}$ ($\frac{1}{m}$)$\geq 1-\frac{1}{m}$. Since $|{\beta_1}^{(m)}|$+ $|{\beta_2}^{(m)}|$ =1, it is clear that 0< $|{\beta_1}^{(m)}|$ <1 and

 DOI : 10.18231/2454-9150.2018.1120

0< $|\beta_2{}^{(m)}|$ <1. Hence we get bounded sequences of real numbers $\{\beta_1{}^{(m)}\}$, $\{\beta_2{}^{(m)}\}$and so there are real numbers γ_1, γ_2such that

$|\gamma_1| + |\gamma_2|$=1, $\beta_1{}^{(m)} \to \gamma_1$ and $\beta_2{}^{(m)} \to \gamma_2$. Consider $y_m = \beta_1{}^{(m)}x_1 + \beta_2{}^{(m)}x_2$. Then $F_{y_m}\left(\frac{1}{m}\right) \geq 1-\frac{1}{m}$. Take y=$\gamma_1 x_1 + \gamma_2 x_2$. Since $F_{y_m}\left(\frac{1}{m}\right)$ $\geq 1-\frac{1}{m}$, $F_{y_m}(t) \geq 1-\frac{1}{m}$ for all t>$\frac{1}{m}$. This implies that F_{y_m} →H. Hence $F_{y_m}(t)$ →1 as m→ ∞ for all t>0. Now F_y=$F_{y-y_m+y_m} \geq \tau(F_{y-y_m}, F_{y_m})$. Since $y_m \to y$ as m→ ∞, we get F_y ≥H . This means that F_y =H and so y=θ. Hence $\gamma_1 x_1 + \gamma_2 x_2$= θ. Since $\{x_1, x_2\}$ are linearly independent $\gamma_1 = \gamma_2 = 0$. Hence $F_{\beta_1 x_1+\beta_2 x_2}$(c)< $1-\delta$ for some c>0, δ ∈(0,1).

Theorem 3.3. Let (X, $\mathcal{F}$,τ) be a probabilistic normed space. Let $\{x_1, x_2, \ldots x_l\}$ be a linearly independent set of vectors. Then there exists c>0, δ ∈(0, 1) such that for any scalars $\alpha_1, \alpha_2, \ldots, \alpha_l$ we have $F_{\alpha_1 x_1+\alpha_2 x_2+\cdots+\alpha_l x_l}$(c $|\alpha_1|$+c $|\alpha_2|$+….+ c $|\alpha_l|$)<1−δ.

Theorem 3.4. Let (X, $\mathcal{F}$,τ) be a probabilistic normed space. Then every finite dimensional probabilistic normed space (X, $\mathcal{F}$,τ) is a complete space.

Proof. Let $\{x_1, x_2, \ldots, x_k\}$ be a basis. Consider $y_n = \beta_1{}^{(n)}x_1 + \beta_2{}^{(n)}x_2+\ldots+\beta_k{}^{(n)}x_k$ where $\beta_i{}^{(n)}$ are scalars. Now $F_{y_n-y_m}$(t)= $F_{(\beta_1{}^{(n)}-\beta_1{}^{(m)})x_1+(\beta_2{}^{(n)}-\beta_2{}^{(m)})x_2+\cdots+(\beta_k{}^{(n)}-\beta_k{}^{(m)})x_k}$(t). By theorem 3.2, there exists c>0, δ ∈(0,1) such that $F_{(\beta_1{}^{(n)}-\beta_1{}^{(m)})x_1+\cdots+(\beta_k{}^{(n)}-\beta_k{}^{(m)})x_k}$ (c($|\beta_1{}^{(n)}-\beta_1{}^{(m)}|$ +…..+ $|\beta_k{}^{(n)}-\beta_k{}^{(m)}|$)) $< 1-\delta$. Suppose $\{y_n\}$ is a Cauchy sequence. Then $F_{y_n-y_m}$(t)=H(t) for all t>0 and n, m→ ∞. Hence $F_{y_n-y_m}(t) > 1-\delta >$ $F_{y_n-y_m}$ (c($|\beta_1{}^{(n)}-\beta_1{}^{(m)}|$+…..+ $|\beta_k{}^{(n)}-\beta_k{}^{(m)}|$)). Since F is non decreasing, c($|\beta_1{}^{(n)}-\beta_1{}^{(m)}|$+…..+ $|\beta_k{}^{(n)}-\beta_k{}^{(m)}|$)<t. Hence$\sum_{i=1}^{k} |\beta_i{}^{(n)}-\beta_i{}^{(m)}| < \frac{t}{c}$. This implies that $|\beta_i{}^{(n)}-\beta_i{}^{(m)}|$ =0 for i=1,2, …., k. Hence for a fixed i, $\{\beta_i{}^{(n)}\}$ is a Cauchy sequence and so $\beta_i{}^{(n)} \to \beta_i$ for some real β_i. Take y=$\beta_1 x_1 + \cdots + \beta_k x_k$. Now F_{y_n-y}(t) = $F_{(\beta_1{}^{(n)}-\beta_1)x_1+\cdots+(\beta_k{}^{(n)}-\beta_k)x_k}$ (t). Hence $\lim_{n\to\infty} F_{y_n-y}$= F_θ =H. This means that y_n →y. Hence (X, $\mathcal{F}$,τ) is a complete space.

Definition 3.5[16]. Let (X, $\mathcal{F}$,τ) be a probabilistic normed space. A subset A of X is said to be bounded if there is a 0< r<1, t>0 such that F_x(t)>1−r for all x ∈A .

Lemma 3.6. Let (X, $\mathcal{F}$,τ) be a probabilistic normed space. A subset A of X is said to be bounded if and only if for each 0< r<1 there is a t>0 such that F_x(t)>1−r for each x ∈A.

Proof. Suppose A is bounded. Hence there is a 0< r<1, t>0 such that F_x(t)>1−r. Take r' such that r<r'<1 then 1−r' <1−r. Since F_x(t)>1−r, we have F_x(t)>1−r' for all x ∈A. Suppose there is a 0<r_1<r and F_x(t)≤ $1-r_1$ for some x ∈A and all t>0. Since F_x(t)→1 as t→ ∞ there is t_1>t such that $F_x(t_1) > 1-r_1$. Which is a contradiction. Conversely suppose for each 0< r<1 there is a t>0 such that F_x(t)>1−r for each x ∈A. Then trivially A is bounded.

Definition 3.7. Let (X, $\mathcal{F}$,τ) be a probabilistic normed space. A subset A of X is said to be closed if for any $\{x_n\}$in A such that $x_n \to x$ implies x ∈A.

A subset A of X is said to be compact if every sequence in A has a convergent subsequence .

Theorem 3.8. Suppose (X, $\mathcal{F}$,τ) is a finite dimensional probabilistic normed space. A subset A of X is compact if and only if A is closed and bounded.

Proof. Suppose A is compact. Let $\{x_n\}$ be a sequence and $x_n \to x$ in A. Since A is compact there exists a subsequence $\{x_{n_k}\}$of $\{x_n\}$ such that $x_{n_k} \to y$ in A. By uniqueness of limit y=x ∈A. Hence A is closed. Now we have to prove that A is bounded. Suppose A is not bounded, then there exists r_0 ∈(0, 1) such that for each n, there exists x_n ∈A such that F_{x_n}(n)≤1−r_0. Consider this sequence $\{x_n\}$. Since A is compact there exists x ∈A such that $x_{n_k} \to x$. That is $F_{x_{n_k}-x}$(t)>1−$\frac{1}{n_k}$ for all t>0. That is $F_{x_{n_k}-x}(n_k)$>1−$\frac{1}{n_k}$. Now $F_{x_{n_k}}(n_k)$ ≤1−r_0. Hence 1−r_0 ≥ $F_{x_{n_k}-x+x}(n_k) \geq \tau(F_{x_{n_k}-x}, F_x)(n_k)$. Since τ is continuous and $x_{n_k} \to x$, 1−r_0 ≥ $\lim_{k\to\infty} F_x(n_k)$=1. This implies that r_0=0, which is a contradiction. Conversely, suppose A is closed and bounded. We claim that A is compact. Let dim X=n and $\{e_1, e_2, \ldots, e_n\}$be a basis. Let $\{x_n\}$ be a sequence in A. Then

 DOI : 10.18231/2454-9150.2018.1120

$x_k = \beta_1{}^{(k)} e_1 + \beta_2{}^{(k)} e_2 + \ldots + \beta_n{}^{(k)} e_n$. By theorem 3.2 there exists c>0, $\delta \in (0,1)$ such that $F_{\beta_1{}^{(k)} e_1 + \beta_2{}^{(k)} e_2 + \cdots + \beta_n{}^{(k)} e_n}(c(\sum_{i=1}^{n} |\beta_i{}^{(k)}|)) < 1-\delta$. Since A is bounded, $F_{\sum_{i=1}^{n} \beta_i{}^{(k)} e_i}(\gamma) > 1-\delta$ for some γ. Hence $F_{\sum_{i=1}^{n} \beta_i{}^{(k)} e_i}(c(\sum_{i=1}^{n} |\beta_i{}^{(k)}|)) < 1-\delta < F_{\sum \beta_i{}^{(k)} e_i}(\gamma)$. Hence $c(\sum_{i=1}^{n}(|\beta_i{}^{(k)}|)) < \gamma$. This implies that $\sum_{i=1}^{n} |\beta_i{}^{(k)}| < \frac{\gamma}{c}$. Hence $\{\beta_i{}^{(k)}\}$ is a bounded sequence in $\mathbb{R}$ for each i. This implies that, for a fixed i for some real number β_i the sequence $\beta_i{}^{(k)} \to \beta_i$. Take $x = \beta_1 e_1 + \cdots + \beta_n e_n$. Now for any t>0 we have $F_{x_{n_k}-x}(t) = F_{\sum_{i=1}^{n}(\beta_i{}^{(k)} - \beta_i) e_i}(t) = F_{(\beta_1{}^{(k)} - \beta_1)e_1 + \cdots + (\beta_n{}^{(k)} - \beta_n)e_n}(t) \geq \tau(F_{(\beta_1{}^{(k)} - \beta_1)e_1}, .., F_{(\beta_n{}^{(k)} - \beta_n)e_n})(t) = 1$. Hence $\lim_{n\to\infty} F_{x_{n_k}-x} = H$ and so $x_{n_k} \to x$. Hence A is compact.

Definition 3.9[15]. Let $(X,\mathcal{F},\tau)$ be a probabilistic normed space. Let $a \in X$, $0<\lambda<1$, $\varepsilon>0$. Define $V(a,\varepsilon,\lambda) = \{x \in X : F_{x-a}(\varepsilon) > 1-\lambda\}$ called (ε,λ) neighborhood of a.

Lemma 3.10. Let $(X,\mathcal{F},\tau)$ be a probabilistic normed space. Let Y, Z be subspaces of X. If $Y \subset Z \subset X$, Y is a proper closed subset of Z. Then for every real number $r \in (0, 1)$ there exists $z \in Z$ such that $F_z(1)>0$ and $F_{z-y}(r)=0$ for all $y \in Y$.

Proof. Let $v \in Z - Y$ and $y \in Y$. Denote $d_y = \inf\{t \in (0,1]: F_{v-y}(t) > 1-t\}$. Suppose $d_y=0$. Then $F_{v-y}(\frac{1}{n}) > 1-\frac{1}{n}$ for all n. Now, for any t>0 there exist n_0 such that $t > \frac{1}{n}$ for all $n \geq n_0$. Hence $F_{v-y}(t) \geq F_{v-y}(\frac{1}{n}) > 1-\frac{1}{n}$ for all $n \geq n_0$. This implies that $F_{v-y}(t)=1$ for any t>0. Hence $F_{v-y}=H$ and so $v=y$, a contradiction. Hence $0<d_y \leq 1$ for any $y \in Y$. Define $d = \inf\{d_y : y \in Y\}$. Then $0 \leq d \leq 1$. If $d=0$ then for all $0<\varepsilon<1$ there exists $y=y(\varepsilon) \in Y$ such that $d_y < \varepsilon$ implies $F_{v-y}(\varepsilon) > 1-\varepsilon$. Hence $V(v, \varepsilon, \varepsilon) \cap Y \neq \emptyset$ for all $0<\varepsilon<1$. Hence every neighborhood of v intersect Y. That is $v \in \bar{Y}$, the closure of Y. Since Y is closed $v \in Y$, which is a contradiction. Hence $0<d \leq 1$. Choose any $r \in (0, 1)$. Then $\frac{d}{r} > d$. Since $d = \inf\{d_y : y \in Y\}$, there is a $y_0 \in Y$ such that $d \leq d_{y_0} < \frac{d}{r}$. Hence there is some $k>0$ such that $d \leq d_{y_0} < k < \frac{d}{r}$. Take $z = \frac{v-y_0}{k}$. Since $k > d_{y_0}$, $F_z(1) = F_{\frac{v-y_0}{k}}(1) = F_{v-y_0}(k) \geq F_{v-y_0}(d_{y_0}) > 1-d_{y_0} > 0$. Hence $F_z(1)>0$. Now for any $y \in Y$ we have

$\inf\{t \in (0,1]: F_{z-y}(t) > 1-t\} = \inf\{t \in (0,1]: F_{\frac{v-y_0}{k}-y}(t) > 1-t\} = \inf\{t \in (0,1]: F_{v-y_0-ky}(kt) > 1-t\} = \inf\{t \in (0,1]: F_{v-(y_0+ky)}(kt) > 1-t\} = \inf\{\frac{s}{k} \in (0,1]: F_{v-(y_0+ky)}(s) > 1-t\} = \frac{1}{k}\inf\{s \in (0,1]: F_{v-(y_0+ky)}(s) > 1-t\} = \frac{1}{k} d_{y_0+ky} \geq \frac{1}{k} d = \frac{d}{k} > r$. Hence $\inf\{t \in (0,1]: F_{z-y}(t) > 1-t\} > r$. This means that $F_{z-y}(r) \leq 1-r$. Since F is non decreasing $F_{z-y}(t) \leq 1-r$ if $t \leq r$. Since $r \in (0, 1)$ is arbitrary $F_{z-y}(t) = 0$ if $t \leq r$. In particular $F_{z-y}(r)=0$, $0<r<1$.

Theorem 3.11. Let $(X,\mathcal{F},\tau)$ be a probabilistic normed space. If the set $M=\{x: F_x(1)>0\}$ is a compact set. Then the space $(X,\mathcal{F},\tau)$ is finite dimensional.

Proof. Suppose dim X=∞. Since $F_\theta(1)=1$, M is a non empty set. Take $x_1 \in X$ such that $F_{x_1}(1)>0$. Consider $X_1=\{x_1\}$, the subspace spanned by x_1. Then X_1 is a closed subspace of X. Hence by lemma 3.9, there exists $x_2 \in X$ such that $F_{x_2}(1)>0$ and $F_{x_2-x_1}(\frac{1}{2})=0$. Consider $X_2=\{x_1, x_2\}$. Since $F_{x_2}(1)>0$ there exists $x_3 \in X$ with $F_{x_3}(1)>0$ and $F_{x_3-x_2}(\frac{1}{2})=0$ and $F_{x_3-x_1}(\frac{1}{2})=0$. Proceeding like this we get a sequence $\{x_n\}$ in X such that $F_{x_n}(1)>0$ and $F_{x_n-x_m}(\frac{1}{2})=0$ if m≠n. By the construction of the sequence $\{x_n\}$, it is clear that neither $\{x_n\}$ nor its subsequence converges. It is a contradiction to $M=\{x: F_x(1)>0\}$ is a compact set. Hence the dimension of X is finite.

V. Conclusion

Finite dimensional normed spaces plays a vital role in Functional Analysis. Here we have studied finite dimensional probabilistic norrmed spaces. This idea throw some light on further development on Fuzzy Analysis.

References

[1] Azad, K.K., On fuzzy semi continuity, fuzzy almost continuity and fuzzy weakly continuity, J1. of Math. Analy. Applns 82 (1981), 14-32.

[2] T.Bag, S.K.Samanta, Finite dimensional fuzzy normed linear spaces, Annals of Fuzzy Mathematics and Informatics, Volume x, No. x,(Month 2013), pp.1-11.

[3] T.Bag, S.K.Samanta, Finite dimensional fuzzy normed linear spaces, The Journal of Fuzzy Mathematics 11(3) (2003)687 – 705.

 DOI : 10.18231/2454-9150.2018.1120

[4] Bernardo Lafuerza Guillen, Jose Antonio Rodriguez Lallena and Carlo Sempi, Completion of Probabilistic Normed Spaces, Internat. J.Math & Math.Sci.Vol.18 No. 4 (1995) 649-652.

[5] Bernardo Lafuerza Guillen, Jose Antonio Rodriguez Lallena and Carlo Sempi, Some classes of Probabilistic Normed Spaces, Rendiconti di Mathematica, serie vii volume 17, Roma(1997),237-252.

[6] Chang, C.L., Fuzzy topological spaces, J. Math. Analy. Applns 24 (1968), 182-190.

[7] Felbin C, Finite dimensional fuzzy normed linear spaces, Fuzzy Sets and Systems 48(1992), 239 – 248.

[8] Erceg, M.A., Metric spaces in fuzzy set theory, J1. of Math. Analy. Applns 69 (1979), 205-230.

[9] Ioan Golet, On probabilistic 2 – normed spaces, NOVI SAD. J. MATH. Vol. 35, No. 1, 2005, 95-102.

[10] Khorasani, A. Abrishami Moghaddam, M.,Best Approximation In Probabilistic 2-Normed Spaces , NOVI SAD. J. MATH. Vol. 40,No.1, 2010,103-110.

[11] Kramosil, I. and Michalek, J., Fuzzy metric and statistical metric spaces, kybernetica 11(1975), 336-344.

[12] K.Menger, Statistical Metrics, Proc. Nat. Acad. Sci., 28(1942), 535-537.

[13]Min-Xian Zhang, Representation theorem on finite dimensional probabilistic nor med space , Scientae mathematicae Japonicae online, vol.10(2004),189-196.

[14] Schweizer, B and Sklar, A., Statistical metric spaces, Pacific J. Maths 10(1960), 313-334.

[15] $\tilde{S}$erstnev A.N., On the notion of random normed space, Dokl. Akad. Nauk SSSR,1963, 49:280-283.

[16] Singadurai. A and Thenmozhi, A., Some Results in Probabilistic Normed Space, The Jl. of Fuzzy Mathematics vol.21, No.3(2013), 659-666.

[17] Tamizh Chelvam, T and Singadurai, A., I-topological vector spaces generated by F-norm, The Jl. of Fuzzy Mathematics vol.14, No.2(2006),255-265.

[18] Zike, Deng, Fuzzy pseudo-metric spaces, Jl. Math. Analy. Applns 86(1982),74-95

[19] Zadeh, L.A. Fuzzy sets, Inform and control 8(1965), 338-353.

 DOI : 10.18231/2454-9150.2018.1120