International Journal for Research in Engineering Application & Management (IJREAM)
ISSN : 2454-9150 Vol-04, Issue-09, Dec 2018

Laplace Transforms Approach for the Velocity Distribution and Shear Stress Distribution of a Unidirectional Laminar Flow

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Abstract - Just as surface tension is the characteristic of a fluid at rest, viscosity is the characteristic of a fluid by virtue of which an internal frictional force (viscous force) produced due to the intermolecular forces becomes active when the fluid is in motion and opposes the relative motion of its different layers. This viscous force acts tangentially and is effective when the different layers of the fluid are moving with different velocities, and leads to a tangential (shearing) stress between the layers of the fluid in motion. In this paper, Laplace transforms approach is applied in obtaining the velocity distribution and the shear stress distribution of a unidirectional laminar flow between the stationary parallel plates as well as between the parallel plates having a relative motion by solving the differential equation describing flow characteristics of the viscous liquid via Laplace transform method. This paper illustrates the application of Laplace transform as a powerful tool to obtain the solution (velocity distribution and shear stress distribution) directly without finding the general solution of a differential equation describing flow characteristics equation of the unidirectional laminar flow between parallel plates.

Index Terms - Laminar flow; Laplace transform; Parallel plates; Shear stress distribution; Velocity distribution; Viscous fluid.

I. INTRODUCTION

The steady flow of a viscous fluid over a horizontal surface in the form of layers of different velocities which do not mix with each other i.e. the flow of a viscous fluid in which the particles of the fluid move in a regular and well - defined paths is known as laminar flow. It appears as if the layers of the fluid slide smoothly over the other layer of the fluid. A velocity gradient exists between the two layers due to relative velocity and as a result, a tangential (shear) stress acts on the layers. Seepage through soils, the flow of crude oil and highly viscous fluids through narrow passages are some of the examples of the laminar flow. In such a flow the fluid properties remain unchanged in the directions perpendicular to the direction of flow of the fluid [1-3].

II. DEFINITION OF LAPLACE TRANSFORMATION

The Laplace transformation of g(y), which is defined for real numbers y ≥ 0, is denoted by G(r) or L {g(y)} and is defined as $L\{g(y)\} = G(r) = \int_0^{\infty} e^{-ry}\, g(y)dy$, provided that the integral exists, i.e. convergent. If the integral is convergent for some value of r, then the Laplace transform of g(y) exists otherwise not, where r *is* the parameter which may be a real or complex number and L is the Laplace transform operator [4-8].

***Laplace Transformation of Elementary* Functions**

1. $L\{1\} = \frac{1}{r}\ ,\ r > 0$

2. $L\{y^n\} = \frac{n!}{r^{n+1}}, r > 0$

$where\ n = 0,1,2, \dots \dots ..$

3. $L\{e^{cy}\} = \frac{1}{r-c}\ ,\ r > c$

4. $L\{sincy\} = \frac{c}{r^2 + c^2}\ ,\ r > 0$

5. $L\{coscy\} = \frac{r}{r^2 + c^2}\ ,\ r > 0$

Proof of Laplace Transformation of Some Functions

1. $L\{1\} = \int_0^{\infty} e^{-ry}\, 1\, dy$

$= \quad -\frac{1}{r}\left(e^{-\infty} - e^{-0}\right) = -\frac{1}{r}(0-1)$

$= \frac{1}{r} = G(r)\ , r > 1$

 DOI : 10.18231/2454-9150.2018.1149

ISSN : 2454-9150 Vol-04, Issue-09, Dec 2018

2. $L\{y^n\} = \int_0^\infty e^{-ry}\, y^n\, dy$

Putting $\quad r\,y = z$ or $y = \frac{z}{r}$ or $dy = \frac{dz}{r}$

We have, $L\{y^n\} = \int_0^\infty e^{-z} \left(\frac{z}{r}\right)^n \frac{dz}{r}$

$$= \frac{1}{r^{n+1}} \int_0^\infty e^{-z} z^n\, dz$$

According to the definition of Gamma function of whole number n, we can write

$$\int_0^\infty e^{-z} z^n\, dz = \overline{|n+1} = n!$$

Hence, $L\{y^n\} = \frac{n!}{r^{n+1}},\ r > 0$

Laplace Transformation of Derivative of a function

If the function g(y), y ≥ 0 is having an exponential order, that is if g(y) is a continuous function and is a piecewise continuous function on any interval, then the Laplace transform of derivative of g(y) [4-8] i.e. $L\{g'(y)\}$ is given by

$$L\{g'(y)\} = \int_0^\infty e^{-ry}\, g'(y) dy$$

Integrating by parts, we get

$L\{g'(y)\} = [0 - g(0)] - \int_0^\infty -re^{-ry}\, g(y) dy,$

Or $L\{g'(y)\} = -g(0) + r\int_0^\infty e^{-ry}\, g(y) dy$

Or $L\{g'(y)\} = rL\{g(y)\} - g(0)$

Or $L\{g'(y)\} = rG(r) - g(0)$

Now, since $L\{g'(y)\} = rL\{g(y)\} - g(0),$

Therefore, $L\{g''(y)\} = rL\{g'(y)\} - g'(0)$

Or $L\{g''(y)\} = r\{rL\{g(y)\} - g(0)\} - g'(0)$

Or $L\{g''(y)\} = r^2 L\{g(y)\} - rg(0) - g'(0)$

Or $L\{g''(y)\} = r^2 G(r) - rg(0) - g'(0)$

And so on.

INVERSE LAPLACE TRANSFORMATION

The inverse Laplace transform of the function G(r) is denoted by L^{-1}[G(r)] or g(y).

If we write L [g(y)] = G(r), then L^{-1}[G(r)] = g(y), where L^{-1} is called the inverse Laplace transform operator [4-8].

Inverse Laplace Transformations of Some Functions

1. $L^{-1}\{\frac{1}{r}\} = 1$
2. $L^{-1}\{\frac{1}{(r-c)}\} = e^{cy}$
3. $L^{-1}\{\frac{1}{r^2+c^2}\} = \frac{1}{c}\sin cy$
4. $L^{-1}\{\frac{r}{r^2+c^2}\} = \cos cy$
5. $L^{-1}\{\frac{1}{r^n}\} = \frac{y^{n-1}}{(n-1)!},\ n > 0$

III. METHODOLOGY

Flow Characteristics Equation

We consider a steady and uniform laminar flow of the viscous fluid between the two flat parallel plates situated at a perpendicular distance L. Let the distance in which the fluid is flowing be represented by x and the distance which is normal to the flow of fluid and parallel to the plane of paper be represented by z such that the lower plate is situated at z = 0 and the upper plate is situated at z = L.

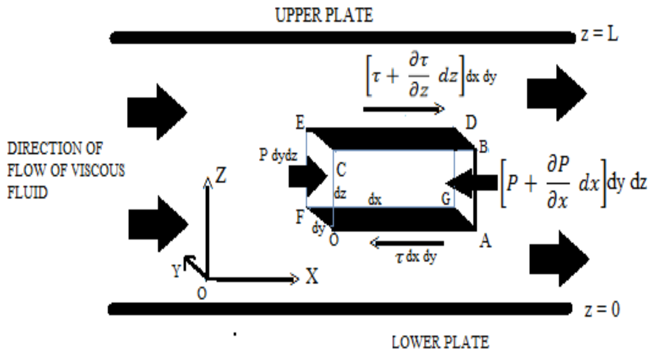

Figure: Flow of viscous fluid between parallel plates.

To derive the flow characteristics equation of motion of a viscous fluid, we consider a small fluid element of length dx, breadth dy and height dz. Due to viscous effects, there exists a relative velocity between any two adjacent layers of the viscous fluid due to which tangential (shear) stress is set up between them. For steady and uniform flow, there will not be any shear stress on the vertical faces of the fluid element [9-11].

If the shear stress on the lower face OAGF of the fluid element is represented by τ and that on the upper face CBDE is $\left[\tau + \frac{\partial \tau}{\partial z} dz\right]$, then the tangential (shearing) force on the fluid element is $\left[\tau + \frac{\partial \tau}{\partial z} dz\right] dx\, dy - \tau\, dx\, dy = \frac{\partial \tau}{\partial z} dx\, dy\, dz$

If the pressure intensity on the face OCEF of the fluid element is represented by P and that on the face ABDG is $\left[P + \frac{\partial P}{\partial x} dx\right]$, then pressure force on the fluid element is $P\, dy\, dz - \left[P + \frac{\partial P}{\partial x} dx\right] dy\, dz = -\frac{\partial P}{\partial x} dx\, dy\, dz$

For steady and uniform flow, the acceleration is zero and hence the sum of the tangential (shearing) force and the pressure force in the direction of flow of the fluid i.e. the resultant force in the x-direction must vanish.

Thus $\frac{\partial \tau}{\partial z} dx\, dy\, dz - \frac{\partial P}{\partial x} dx\, dy\, dz = 0$

Or $\quad \frac{\partial \tau}{\partial z} = \frac{\partial P}{\partial x} \quad \ldots..(1)$

Since we are concentrating on pressure gradient only in the direction of flow of fluid, we can replace the partial derivatives by total derivatives. Thus

 DOI : 10.18231/2454-9150.2018.1149

ISSN : 2454-9150 Vol-04, Issue-09, Dec 2018

$\frac{d\tau}{dz} = \frac{dP}{dx}$ ……… (2)

This means that the shearing gradient in the direction normal to the flow of the viscous fluid is equal to the pressure gradient along the direction of flow of the fluid. According to Newton's law of viscosity, the value of shear stress is given by

$\tau = \mu \dot{U}(z)$ ………… (3)

$\cdot \equiv \frac{d}{dz}$

Where $\dot{U}(z)$ represents the rate of change of velocity w.r.t. z i.e. it represents the velocity gradient in the direction normal to the flow of fluid, and μ represents the coefficient of viscosity.

Put equation (3) in equation (2), we get

$\mu \ddot{U}(z) = \frac{dP}{dx}$ ……….. (4)

Solution of the Flow Characteristics Equation

We will analyze the laminar flow of viscous fluid on the basis of following assumptions [1-3]:

i) The flow is steady and incompressible and the properties of the fluid do not vary in the directions normal to the direction of flow of the fluid.

ii) There are no end effects of the surfaces on the viscous fluid.

iii) There is a uniform effective pressure gradient in the direction of flow of fluid i.e. $\frac{dP}{dx}$ is a constant in the x–direction.

iv) There is no relative velocity of the fluid with respect to the surfaces of the plates.

Now taking Laplace Transform of equation (4), we get

$L[\mu \ddot{U}(z)] = \frac{dP}{dx} L[1]$

This equation results

$\mu[r^2\bar{U}(\mathrm{r}) - rU(0) - \dot{U}(0)] = \frac{1}{r}\frac{dP}{dx}$ ……… (5)

(a) For Laminar Flow Between Stationary (Fixed) Parallel Plates

Considering the flow of fluid between two parallel fixed plates, we can write the relevant boundary conditions as given below [9-11]:

At z = 0 and z = L, $U = 0$.

Applying boundary condition: $U(0) = 0$, equation (5) becomes,

$\mu[r^2\bar{U}(\mathrm{r}) - \dot{U}(0)] = \frac{1}{r}\frac{dP}{dx}$ ….. (6)

In this equation, $\dot{U}(0)$ is some constant so let us substitute $\dot{U}(0) = \varepsilon$. Also, since $\frac{dP}{dx}$ is uniform, therefore, put $\frac{dP}{dx} = -\emptyset$, where $\emptyset$ is a constant and negative sign indicates that the pressure of fluid decreases in the direction of flow of the fluid.

Equation (3) becomes

$\mu[r^2\bar{U}(\mathrm{r}) - \varepsilon] = \frac{-\emptyset}{r}$

Or $\bar{U}(\mathrm{r}) = \frac{\varepsilon}{r^2} - \frac{\emptyset}{\mu r^3}$………. (7)

Taking inverse Laplace transform of equation (7), we get

$U(\mathrm{z}) = \varepsilon\, \mathrm{z} - \frac{\emptyset}{2\mu} z^2$……….. (8)

Determination of the Constant ε

To find the value of constant ε, applying boundary condition: $U(L) = 0$, equation (8) provides,

$0 = \varepsilon\, \mathrm{L} - \frac{\emptyset}{2\mu} L^2$

Upon rearranging and simplification of the above equation, we get

$\varepsilon = \frac{\emptyset}{2\mu} L$ ………. (9)

Substitute the value of ε from equation (9) in equation (8), we get

$U(\mathrm{z}) = \frac{\emptyset}{2\mu} L\, \mathrm{z} - \frac{\emptyset}{2\mu} z^2$

Or $U(\mathrm{z}) = \frac{\emptyset}{2\mu}[L\, \mathrm{z} - z^2]$……… (10)

Differentiating equation (10) w.r.t. z, we get

$\dot{U}(z) = \frac{\emptyset}{2\mu}[L - 2z]$ …….. (11)

For maximum velocity, $\dot{U}(z) = 0$

This results

$\mathrm{z} = \frac{L}{2}$ ……………. (12)

Put the value of z from equation (12) in equation (10), we get

$$U_{max} = \frac{\emptyset}{2\mu}\frac{L^2}{4}$$

Or

$U_{max} = \frac{\emptyset}{8\mu} L^2$ ………. (13)

The equations (10) and (12) confirm that the velocity distribution is maximum at the midway between the stationary (fixed) parallel plates and decreases parabolically with a maximum value at the midway between the fixed parallel plates to a minimum value at the lower fixed plate as well as at the upper fixed plate.

The shear stress distribution is determined by the application of Newton's law of viscosity as

$\tau(\mathrm{z}) = \mu \dot{U}(z)$

DOI : 10.18231/2454-9150.2018.1149

Using equation (8), we get

$\tau(z) = \frac{\emptyset}{2}[L - 2z]$.......... (14)

At z $= \frac{L}{2}$ i.e. at the mid of the fixed parallel plates, $\tau\left(\frac{L}{2}\right) = \frac{\emptyset}{2}[L - 2\frac{L}{2}] = 0$ i.e. there is no shear stress even when there is constant pressure gradient.

At z = 0 i.e. at the surface of the lower fixed plate, $\tau(0) = \frac{\emptyset}{2}L$

At z $= L$ i.e. at the surface of the upper fixed plate, $\tau(L) = -\frac{\emptyset}{2}L$

For a particular case, when $\emptyset = 0, \tau(z) = 0$ i.e. there is no shear stress between the fixed parallel plates if there is no pressure gradient.

The equation (14) confirms that the shear stress varies linearly in the presence of constant pressure gradient between the fixed parallel plates with a minimum value at the midway between the fixed parallel plates to a maximum value at the lower fixed plate as well as at the upper fixed plate.

(b) For Laminar Flow Between Parallel Plates Having Relative Motion

Considering the flow of fluid between the parallel flat plates such that the lower plate is fixed at z = 0 and upper plate is moving uniformly with velocity U_o relative to the lower fixed plate in the direction of flow of the fluid, we can write the relevant boundary conditions as given below [9-11]:

At z = 0, $U = 0$ and at z = L, $U = U_o$.

Applying boundary condition: $U(0) = 0$, equation (5) becomes,

$\mu[r^2\overline{U}(\mathrm{r}) - \dot{U}(0)] = \frac{-\emptyset}{r}$ ..... (15)

In this equation, $\dot{U}(0)$ is some constant.

Let us substitute $\dot{U}(0) = \delta,$

Equation (15) becomes

$$\mu[r^2\overline{U}(\mathrm{r}) - \delta] = \frac{-\emptyset}{r}$$

Or $\overline{U}(\mathrm{r}) = \frac{\delta}{r^2} - \frac{\emptyset}{\mu r^3}$.......... (16)

Taking inverse Laplace transform of equation (16), we get

$U(\mathrm{y}) = \delta\,\mathrm{z} - \frac{\emptyset}{2\mu}z^2$.......... (17)

Determination of the Constant δ

To find the value of constant δ, applying boundary condition: $U(L) = U_o$, equation (17) provides,

$$U_o = \delta\,\mathrm{L} - \frac{\emptyset}{2\mu}L^2$$

Upon rearranging and simplification of the above equation, we get

$\delta = \frac{U_o}{L} + \frac{\emptyset}{2\mu}L$ .... (18)

Substitute the value of δ from equation (18) in equation (17), we get

$$U(\mathrm{z}) = [\frac{U_o}{L} + \frac{\emptyset}{2\mu}L]\,\mathrm{z} - \frac{\emptyset}{2\mu}z^2$$

Or $U(\mathrm{z}) = \frac{U_o}{L}z + \frac{\emptyset}{2\mu}[L\,\mathrm{z} - z^2]$ ... (19)

This equation (19) confirms that the velocity distribution is parabolic with minimum at the lower fixed plate.

Differentiating equation (19) w.r.t. z, we get

$\dot{U}(z) = \frac{U_o}{L} + \frac{\emptyset}{2\mu}[L - 2z]$...... (20)

For maximum velocity, $\dot{U}(z) = 0$

This results

z $= \frac{L}{2} - \frac{\mu U_o}{L\emptyset}$................ (21)

Put the value of z from equation (21) in equation (19) and simplifying, we get

$U_{max} = \frac{\mu U_o{}^2}{L^2\,\emptyset}$......... (22)

The shear stress distribution is determined by the application of Newton's law of viscosity as

$\tau(z) = \mu\,\dot{U}(z)$

Using equation (20), we get

$\tau(z) = \frac{\mu U_o}{L} + \frac{\emptyset}{2}[L - 2z]$...... (23)

At z $= \frac{L}{2}$ i.e. at the mid of the flow passage, $\tau\left(\frac{L}{2}\right) = \frac{\mu U_o}{L}$

At z = 0 i.e. at the surface of the lower plate, $\tau(0) = \frac{\mu U_o}{L} + \frac{\emptyset}{2}L$

At z $= L$ i.e. at the surface of the upper plate, $\tau(L) = \frac{\mu U_o}{L} - \frac{\emptyset}{2}L$

For a particular case, when $\emptyset = 0$, $\tau(z) = \frac{\mu U_o}{L}$ i.e. the shear stress between the plates is not zero and having a constant value even if there is no pressure gradient.

The equation (23) confirms that the shear stress varies linearly in the presence of constant pressure gradient between the parallel plates having relative motion, and at the midway between the parallel plates it is equal to the mean of the values of the shear stresses at the lower fixed plate and the uniformly moving upper plate.

IV. CONCLUSION

In this paper, we have obtained the velocity distribution and the shear stress distribution of a unidirectional laminar flow between stationary parallel plates as well as laminar flow

 DOI : 10.18231/2454-9150.2018.1149

between parallel plates having a relative motion by solving the differential equation describing the flow characteristics of a viscous fluid via Laplace transform method. Thus, this method is an interesting approach and a novelty and illustrates the Laplace transform as a powerful tool in obtaining the solution (velocity distribution) of the differential equation representing flow characteristics without finding the general solution. We concluded that in the case of laminar flow with constant pressure gradient between stationary parallel plates, the velocity distribution is maximum at the midway between the parallel plates and decreases parabolically with maximum value at the midway between the parallel plates to a minimum value at the lower fixed plate as well as at the upper fixed plate but the shear stress varies linearly with a minimum value at the midway between the parallel plates to a maximum value at the lower fixed plate as well as at the upper fixed plate. In the case of laminar flow with constant pressure gradient between parallel plates having a relative motion, the velocity distribution is parabolic with a minimum at the lower fixed plate but the shear stress varies linearly and at the midway between the parallel plates having a relative motion it is equal to the mean of the values of the shear stresses at the lower fixed plate and at the uniformly moving upper plate, and having a constant value even if there is no pressure gradient between parallel plates having a relative motion.

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