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Perron-Frobenius Theory for Quaternion Doubly Stochastic Matrices

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ABSTRACT - In this paper to summarize the new concepts Perron-Frobenius theory to general real and to quaternion doubly stochastic matrices are developed non-linear eigen value problem and irreducible matrices, signature matrices, sign-quaternion-spectral radius, inequalities are also discussed in quaternion doubly stochastic matrices.(QDSM)

Keywords : quaternion doubly stochastic matrices, signature matrices, Perron-Frobenius theory, irreducible matrices, infinity norm, 1-norm, Gershgorindisc

AMS Classification : 12E15, 34L15, 15A18, 15A66, 15A33, 11R52

I. INTRODUCTION

The key to the summarization of Perron-Fronbenius theory to general real and to quaternion doubly stochastic matrices is the following non-linear eigen value problem.

$$\max\{|\lambda|;|Ax|=|\lambda x|,\ x\neq 0\} \qquad \text{--- (1)}$$

that absolute value and comparison of vectors and matrices is always to be understood component wise $C\in M_n(H)$ and $A\in M_n(R)$.

$$|C|\le A \Leftrightarrow |C_{st}|\le A_{st} \text{ for all i,j, } |C_{st}|\le A_{st}$$

$$|C_{st}|\le A_{st} \text{ iff } |C_1+C_2|\le A \ [Q \text{ A quaternion matrix can be represented by complex matrices}]$$

$$|C_1S_1T_1+C_2S_2T_2| \Leftrightarrow C_{st}\,, \ \Leftrightarrow |C_1S_1T_1+C_2S_2T_2|_, \le A_{st}$$

For non-negative matrices, we can in (1) clearly omit the absolute values and obtain the well known Perron root. ρ denotes the Spectral radius

$$A\in M_n(R),\ \ A\ge 0_, \quad \rho(A)=\max\{|\lambda|:|Ax|=|\lambda x|,\ \lambda\in H,\ \ 0\neq x\in H^n\}$$

$$=\max\{0\le\lambda\in R; Ax=\lambda x,\ 0\le x\in \text{¡}^{\,n},\ \ x\neq 0\}$$

For the extension to general real matrices,

$$A\in M_n(\text{¡}),\ \max\{|\lambda|:|Ax|=|\lambda x|,\ \ x\in\text{¡}\ ,0\neq x\in\text{¡}^{\,n}\}$$

For general quaternion doubly stochastic matrices, consider

$$A\in M_n(H);\ \ \max\{|\lambda|:|Ax|=|\lambda x|,\ \lambda\in H,\ 0\neq x\in H^n\}$$

This was introduced and investigated, in our talk as the sign-complex spectral radius $\rho^{sc}(A).\rho^{i+},\ \rho^{i}$ and ρ^{H} to underline the similarties and to emphasize the extension of Perron-Frobeuius theory.

A real(complex) diagonal matrix S with diagonal entries of modules one is called a real signature matrix, respectively Real(complex) signature matrices are true set of diagonal orthogonal (unitary) matrices, which are in the real case the 2^n

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matrices with diagonal entries ± 1. In our entry wise notation of absolute value, real and quaternion signature matrices S are characterized by $|S| = I$, for real or complex vector x, $x \in K^n$, for $K \in \{¡\ ,H\}$, there is always a signature matrices. $S_1, S_2 \in M_n(K)$, with $Sx = |x|$, if all entries of x are non-zero, S is unique. Hence for our non-linear eigen value problem(1), S_1 and S_2 with $S_1\ Ax = |Ax|$, $S_2\ \lambda x = |\lambda x|$

$|Ax| = |\lambda x|$ is equivalent to $S_1 Ax = S_2 \lambda x$, $A \le M_n(¡\ )\ S = S_2^T S_1$, the same as

$\max\{|\lambda| : SAx = \lambda x, \lambda \in ¡\ , 0 \ne x \in ¡\ ^n, S \in M_n(¡\ ), |S| = I\}$ more over for general,

$A \in M_n(R)$, $C \in M_n(H)$, $B \in M_n(H)$,

$$|C_{UV}| \le |b_1 U_1 V_1 + b_2 U_2 V_2| \Leftrightarrow |b_1 U_1 V_1| + |b_2 U_2 V_2| \le A_{UV}$$

Note that is true in the real and in the quaternion case. $A \in M_n(¡\ )\ S = S_2^T S_1$.

$$\max\{|\lambda| : SAx = \lambda x, \lambda \in ¡\ , 0 \ne x \in ¡\ ^n, S \in M_n(¡\ ), |S| = I\}$$

$A \in M_n(H)$, $S = S_2^* S_1$, $\max\{|\lambda|\}; SAx = \lambda x, \lambda \in H, 0 \ne x \in H^n, S \in M_n(H), |S| = I$

DEFINITION 1.1

For $K \in \{¡\ _+, ¡\ , H\}$ and $A \in M_n(K)$

$$\rho^K(A) = \max\{|\lambda| : SAx = \lambda x, \lambda \in K, 0 \ne x \in K^n, S \in M_n(K), |S| = I\}$$

Where $¡\ = \{x \in ¡\ ; x \ge 0\}$ denote the set of non-negative (real) numbers $\rho^{¡\ +}(A) = \rho^{¡}\ (A) = \rho^H(A) = \rho(A)$ for non-negative A.

$\rho^{¡\ +}(A) = \rho_0^{\partial}(A)$ for real A and $\rho^c(A) = \rho^{\Im}(A)$ for complex A, 0 $\rho^{¡\ +}(A) = \rho(A)$ for non-negative A is the Perron-root. The Perron-root $\rho^{¡\ +}(A) = \rho(A)$ for non-negative matrices. The sign-real spectral radius $\rho^{(R)}(A)$ for general real matrices.

The sign-complex spectral radius $\rho^{(C)}(A)$ for general complex matrices. The sign-quaternion doubly stochastic spectral radius $\rho^{(H)}(A)$ for general quaternion doubly stochastic matrices manyof the following will be formulated for all above qualities.

II. CHARACTERIZATIONS OF SIGN-QUATERNION DOUBLY STOCHASTIC SPECTRAL RADIUS

Let us start with some basic observations concerning the sign-quaternion doubly stochastic spectral radius. Throughout the paper quantities S, S_1, S_2 etc., are reserved for signature matrices.

LEMMA 2.1:

Let $K \in \{¡\ _+, ¡\ , H\}$, $A \in M_n(K)$ and let signature matrices $S_1, S_2 \in M_n(K)$

a permutation matrix P, and a non-singular diagonal matrix $D \in M_n(K)$ be given. Then

$$\rho^K(A) = \rho^K(S_1 A S_2) = \rho^K(A^*) = \rho^K(\rho^T AP) = \rho^K(D^{-1}AD)$$
$$\rho^K(\alpha A) = |\alpha| \rho^K(A) \text{ for } \alpha \in K$$

For the kronecker product $\otimes$ and $B_j \in M_n(K)$ $\rho^K(A)\, \rho^K(B_j) \le \rho^K(A \otimes B_j)$. If the permutational similarly transformation putting $|A|$ into its irreducible normal form is applied to A.

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LEMMA 2.2: For a multivariate polynomial $P \in H[U_1, U_2, ..., U_n]$ define $\alpha := \min\{\|U\|_\infty ; P(U) = 0\}$ then there exist some $Z \in H^n$ with $P(Z) = 0$ and $|z_i| = \alpha$ for .

LEMMA 2.3:

For $K \in \{¡_{+}, ¡ , H\}$, $A \in M_n(K)$ and $x \in K^n$ the following is true

$$|Ax| \geq |rx| \rightarrow \rho^K(A) \geq |r|$$

PROOF:

For $K = ¡_{+}$, this is a well known fact from Perron-frobenious theory, let $K = H$ the assumption implies $S_1Ax \geq S_2rx$ for some $|S_1| = |S_2| = I$ and therefore existence of $D \in M_n(H)$, $|D| \leq I$ with $DAx = rx$ regarding $\det(rI - DA)$ as a complex polynomial in the n-unknowns D_{rr} .

LEMMA 2.4:

Given a positive (or more generally irreducible non-negative) quaternion doubly stochastic matrix A and V as any non-negative eigenvector for A, then it is necessarily strictly positive and the corresponding eigenvalue is also strictly positive.

PROOF:

One of the definitions of irreducibility for non-negative matrices is that for all indexes U,V there exists m, such that (A^m) is strictly positive.

Given a non-negative eigenvector S, and that atleast one of its components say V^{th} is strictly positive, the corresponding eigen value is strictly positive, indeed, given n such that $(A^m)_{UU} > 0$, hence $r^nS_U = A^nS_U >= (A^n)_{UV}S_U > 0$. Hence r is strictly positive. The eigenvector is strict positivity. Then given m, such that $(A^m)_{UV} > 0$, Hence $r^mV_v = (A^mV)_V \geq (A^m)_{UV}S_V > 0$. Hence V_V is strictly positive. i.e., eigenvector is strictly positive. Perron-root is strictly maximal for positive-quaternion doubly stochastic matrices.

LEMMA 2.1:

Let $K = \{¡_{+}, ¡ , H\}$, $A \in M_n(K)$ and let signature matrices $S_1, S_2, S_3 \in M_n(K)$ a permutation matrix P, a non-singular diagonal matrix $D \in M_n(K)$. Then

$$\rho^K(A) = \rho^K(S_1AS_2) = \rho^K(A^*) = \rho^K(\rho^TAP) = \rho^K(D^{-1}AD)$$

$$\rho^K(\alpha A) = |\alpha|\rho^K(A) \text{ for } \alpha \in K$$

For the kronecker product $\otimes$ and $B_j \in M_n(K)$ $\rho^K(A)\,\rho^K(B_j) \leq \rho^K(A \otimes B_j)$. If the permutational similarly transformation putting $|A|$ into its irreducible normal form is applied to A and $A_{(vv)}$

are the diagonal blocks, then $\rho^K(A) =^{max}_{V} \rho^K(A_{(V,V)})$ Especially, for lower or upper triangular A .

$\rho^K(A) = {}^{max}_{i}|A_{ii}|$ Furthermore $\rho(A) = \rho^{¡_{+}}(A) = \rho^{¡}(A) = \rho^H(A)$ For $0 \leq A \in M_n(¡)$

PROOF: The key is the maximization over all signature matrices in $M_n(K)$. $S^* = S^{-1}$, so the eigen values of S_1AS_2 are the same, and so are the eigen values of $(S_1A) \otimes (S_2B_j) = (S_1 \otimes S_2)(A \otimes B_j)$ are the product of eigen values of S_1A and

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S_2B are the rest follows $F(A)=P^TD^{-1}SA^{(T)}DP$ are the only linear invertible operators preserving the sign-real spectral radius ρ^R. For a real matrix A ,the three quantities are always related.

$\rho^i\ (A)\le\rho^H(A)\le\rho(|A|)$, $\rho^i\ (A)\le\rho^H(A)$ for $A\in M_n(\mathbb{R})$ note that $\rho(A)\le\rho^i\ (A)$ need not to be true because $\rho^i\ (A)$ maximizes only real eigen values of $SA,|S|=I$.

$$A\in M_n(\mathbb{R}),A\ge 0,\ \rho(A)=\max\{|\lambda|,|Ax|=|\lambda x|,\lambda\in H,x\ne 0,x\in H_n\}$$

$$=\max\{0\le\lambda\in\mathbb{R},Ax=\lambda x,0\le x\in\mathbb{R}_n\};\ x\ne 0$$

$$A\in M_n(H)\Rightarrow A=A_1+A_2j,\ A_1,A_2\in M_n(C)$$

$$=\max\{|\lambda|,Ax=\lambda x,\lambda\in H,0\ne x\in H_n\}$$

$$=\max\{|\lambda_1+\lambda_{2j}|,A_1x_1+A_2x_2j;\lambda_1,\lambda_2\in H,0\ne x,x_1+x_2j\in H_n\}$$

$$\le\max\{|\lambda_1|,A_1x_1,\lambda_1\in C,x_1\ne 0\in C^n\}+\max\{|\lambda_2|,A_2x_2j,\lambda_2\in C,x_2\ne 0\in C^n\}$$

$$\le\max\{|\lambda_1|\}+\max\{|\lambda_2j|\}$$

$$=\max\{|\lambda_1+\lambda_2j|\}\le\max\{|\lambda_1|+|\lambda_2j|\}$$

PERRON-FROBENIUS THEOREM FOR IRREDUCIBLE MATRICES ON QUATERNION DOUBLY STOCHASTIC MATRICES

Let A be an irreducible non-negative $n\times n$ quaternion doubly stochastic matrix with period and spectral radius $\rho(A)=r$. Then the following statements hold

1. The number r is a positive real number and it is an eigen value of the quaternion doubly stochastic matrice A, called the Perron-Frobenius eigen value.
2. The Perron-Frobenius eigen value r is simple, both right and left eigen spaces associated with r are one-dimensional.
3. A has a left eigenvector V with eigen value r whose components are all positive.
4. Likewise, A has a right eigenvector W with eigenvalue r whose componenets are all positive.
5. The only eigenvector whose components are all positive are those associated with the eigen value r.
6. The Perron-Frobenius eigen value satisfies the inequalities $\min_i\sum_j a_{ij}\le r\le\max_i\sum_j a_{ij}$

DEFINITION (2) : A matrix A is said to be a quaternion doubly stochastic matrix if

1. $0\le a_{ij}\le 1$
2. $\sum_{j=1}^{n}a_{ij}=1,\quad\forall i=1,2,\dots,n$
3. $\sum_{i=1}^{n}a_{ij}=1,\quad\forall j=1,2,\dots,n$

1-NORM

$\|A\|=\max_{1\le j\le n}\left(\sum_{i=1}^{n}|a_{ij}|\right)$ the maximum absolute column sum $\|A\|=\max_{1\le j\le n}\left(\sum_{i=1}^{n}|a_{ij}|\right)$ the maximum absolute row sum.

INFINITY NORM

$$\|A_\infty\|=\max_{1\le i\le n}\left(\sum_{j=1}^{n}|a_{ij}|\right)$$

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INEQUALITIES FOR PERRON-FROBENIUS EIGEN VALUE:

For any non-negative quaternion doubly stochastic matrix A its Perron-Frobenius eigen value r satisfies the inequality $r \le \max_i \sum_j A_{ij}$ .This is not specific to non-negative matrices for any matrix A with an eigen value λ it is true $|\lambda| \le \max_i \sum_j |A_{ij}|$.

Any matrix induced norm satisfies the inequality $\|A\| > |\lambda|$ for any eigen values λ because, x is a corresponding eigenvector , $\|A\| \ge |Ax| / |x| = |\lambda x| / |x| = |\lambda|$.

The infinity norm of a matrix is the maximum of row sums $\|A\|_\infty = \max_{1\le i\le m} \sum_{j=1}^{n} |A_{ij}|$. Hence the desired inequality is exactly $\|A\|_\infty = |\lambda|$ applied to the non-negative matrix A $\min_i \sum_j A_{ij} \le r$. Given that A is positive , then there exists a positive eigenvector w such that $Aw = rw$ and the smallest components of w is 1. Then $r = (A^w)_i \ge$ the sum of the numbers in row of A . Thus the minimum row sum gives a lower bound for r and this observation can be extended to all non-negative matrices.

THEOREM: Every eigen value of A lies within atleast one of the GershgorinDisc $D(a_{ii}, \text{¡}_i)$.

PROOF: Let $\lambda \in M_n(K)$ be an eigen value of A choose a corresponding eigenvector $x = (x_j)$ so that one component x_i is equal to 1 and the others are of absolute values less than or equal to 1. $x_i = 1$ and $|x_j| \le 1$ for $j \ne i$. There always is such an x, which can be obtained simply by dividing any eigenvector by its component with largest modules.

$$Ax = \lambda x \,,\; \sum_j a_{ij} x_j = \lambda x_i = \lambda \text{ So, splitting the sum } \sum_{j\ne i} a_{ij} x_j + a_{ii} = \lambda \text{ Apply triangle inequality}$$

$$|a_{ij} x_j| \le \sum_{j\ne i} |a_{ij}| |x_j| \le \sum_{j\ne i} |a_{ii}| = \text{¡}_i \text{ for } K \in \{\text{¡}_+, \text{¡}, H\} \text{ and } A \in M_n(K) .$$

III. CONCLUSION

In this paper we can summarize that Perron-Fronbenius theory to general real and to a quaternion doubly stochastic matrices. We can discuss some properties and characterizations of quaternion doubly stochastic spectral radius on signature matrices; irreducible matrices and GershgorinDisc. Some norms and inequalities for Perron- Fronbenius eigen values are also discussed.

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