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Chebyshev Collocation Method for Singular Integro-Differential Equations with Cauchy Kernel

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Abstract: In this paper, we present a new application of Chebyshev collocation method for solving the singular integro-differential equation with Cauchy kernel. We use truncated Taylor series polynomial of the unknown function and transforms the integro-differential equation to nth order differential equation with variable co-efficients. The resulting differential equation is then solved by using proposed collocation method. Some numerical examples are given to illustrate the efficiency and accuracy of the method.

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Keywords: Singular Integro-Differential Equation, CauchyKernel, Chebyshev polynomial.

I. INTRODUCTION

Integro-differential equation is an equation that involves both integrals and derivative of unknown function. In recent years, many researchers and scientists studied the Integro-differential equation through their work in scientific applications like heat transformer,neutron diffusion and biological species co-existing together with increasing and decreasing rates of generating and diffusion process in general. These kind of equations can also found in physics,biology and engineering applications as well as in models dealing with advanced integral equations [1, 2, 3, 4]. Integro-differential Equation with singular kernels appear in studies involving elastic contact[5],stress analysis, fracture mechanics[6, 7, 8], airfoil theory [9, 10, 11, 12, 13, 14] ,combined infrared reaction and molecular conduction [15, 16]. Cauchy singular integro-differential equation [17, 18] was introduced early in the 20^{th} century, which has undergone an intense growth. The study of singular integro-differential equation with Cauchy kernel have great interest in contemporary research work in which several numerical methods developed and applied to obtain the approximate solutions. Sankar et.al. [5] worked on power series formulation instead of orthogonal polynomials and subsequently collocation method for singular integro-differential equations. Ioakimidis[19] solved singular integral equations by collocation method. Frankel[20] proposed Galerkin solution to a regularized singular integro-differential equation with Cauchy kernel. Badar [21] presented a Galerkin approach solving linear integro-differential equation with Cauchy kernel using orthogonal basis functions Legendre polynomial of the first and second kind. Arzang [22] worked on Galerkin method for fredhlom Singular integro-differential equation. Later Bhattacharya and Mandal [23] proposed a method based on polynomial approximation using Bernstein polynomial basis. Recently Abdelaziz Mennouni [24] worked on the iterated projection method by using Legendre polynomials.

In this paper we consider integro-differential equation of the form

$$\mu(x)\frac{d\phi(x)}{dx} + \lambda(x)\int_{-1}^{1} k(x,y)dy = f(x) \qquad (1)$$

where $f(x), k(x,y), \mu(x), \lambda(x)$ are given functions and $\phi(x)$ is the unknown. We suppose that $\phi(x) \in C^n[-1,1]$, $k(x,y) \in C^n[-1,1]$ and $\mu(x), \lambda(x), f(x) \in L_2[-1,1]$. Here the kernel k(x,y) has the forms

$$k(x,y) = \frac{1}{x-y}\sum_{i=0}^{n} m_i(x)n_i(y) \text{ or } k(x,y) = h(x,y)\ln|x-y|$$

where $m_i(x), n_i(x), h(x,y) \in C^n[-1,1]$.

In the present work, we consider the integro-differential equation with cauchy kernel in the form

$$\mu(x)\frac{d\phi(x)}{dx} + \lambda(x)\int_{-1}^{1} \frac{\phi(y)}{y-x}dy = f(x) \qquad (2)$$

II. CONVERSION OF INTEGRO-DIFFERENTIAL EQUATION TO DIFFERENTIAL EQUATION

We consider the integro-differential equation given Eq. 2 under the given condition in Eq.1. Suppose $\phi(y)$ satisfies the Taylor series theorem condition. We convert the integro-differential equation 2 to a differential equation by using Taylor series expansion of $\phi(y)$ based on expanding about the given point x∈[-1,1] [22].

$$\phi(y) = \phi(x) + (y-x)\phi'(x) + \frac{(y-x)^2}{2!}\phi''(x) + \ldots + \frac{(y-x)^n}{n!}\phi^n(x) \qquad (3)$$

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By substituting relation 3 into Eq.2 we have

$$\mu(x)\phi'(x)+\lambda(x)\left[\int_{-1}^{1}\frac{1}{y-x}\left\{\phi(x)+(y-x)\phi'(x)+\frac{(y-x)^2}{2!}\phi''(x)+\ldots+\frac{(y-x)^n}{n!}\phi^n(x)\right\}\right]dy=f(x) \quad (4)$$

In order to simplify Eq.4 We compute the following integrals:

$$\int_{-1}^{1}\frac{\phi(x)}{y-x}dy=\int_{-1}^{1}\frac{\phi(x)}{y-x}dy+\int_{-1}^{1}\phi'(x)dy+\int_{-1}^{1}\frac{y-x}{2!}\phi''(x)dy+\ldots\ldots+\int_{-1}^{1}\frac{(y-x)^{(n-1)}}{n!}\phi^n(x)dy \quad (5)$$

under the condition $|x|<1$ we have

$$\int_{-1}^{1}\frac{\phi(x)}{y-x}dy=\phi(x)\ln|y-x|_{y=-1}^{y=1}=\phi(x)\ln\frac{(1-x)}{(1+x)}$$

The Taylor series of $\ln(\frac{1-x}{1+x})$ would be

$$\ln(\frac{1-x}{1+x})=2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\ldots\ldots\right)$$

and for every $n\in N$

$$\int_{-1}^{1}\frac{(y-x)^{(n-1)}}{n!}dy=\frac{1}{nn!}\left[(1-x)^n+(-1)^{(n+1)}(1+x)^n\right]$$

Under above computation Eq.2 transforms into the following nth order differential equation with variable coefficients:

$$\lambda(x)\ln\left(\frac{1-x}{1+x}\right)\phi(x)+(\mu(x)+2\lambda(x))\phi'(x)+\lambda(x)\sum_{k=2}^{k=n}a_k(x)\phi^{(k)}(x)=f(x) \quad (6)$$

where

$$a_k(x)=\frac{1}{kk!}\left[(1-x)^k+(-1)^{(k+1)}(1+x)^k\right], k=2,3,\ldots,n \quad (7)$$

III. DESCRIPTION OF THE PROPOSED METHOD

In this section, we discuss the method applied to solve the differential equation obtained in section 2.

Definition 3.1 Chebyshev polynomial of order n is denoted and defined by

$$T_n(x)=\cos(n\cos^{-1}x)\ \text{, where } T_0(x)=1, T_1(x)=x$$

which may be generated from the Rodrigues formula: $T_n(x)=\frac{(-2)^n n!\sqrt[2]{1-x^2}}{2n!}\frac{d^n}{dx^n}(1-x^2)^{n-1/2}$

In this section, we have presented a collocation method based on chebyshev polynomial to solve ordinary differential equation. To describe the method, we consider the general nth order ordinary differential equation is of the form:

$$y^{(n)}(t)=f\left(t,y,y',y'',\ldots y^{(n-1)}\right) \quad (8)$$

With the initial conditions

$$y^{(i)}(t_0)=\propto_i,\quad i=0,1,2,3,\ldots(n-1) \quad (9)$$

Where $\propto_i$, s are constants. Let us assume that,

$$y^{(n)}(t)\approx B^H\ T(t)=\sum_{i=0}^{n}c_iT_i(t) \quad (10)$$

Where,

$$B^H=(c_0,c_1,\ldots\ldots c_n)$$

$$T(t)=(T_0(t),\ldots\ldots T_n(t))$$

Now, Chebyshev polynomial coefficients to be determined. Integrating Eq.10 with respect to t from t_0 to t, n times yeilds.

$$\begin{aligned}
y^{(n-1)}(t)&=y(t_0)+\int_{t_0}^{t}B^H\ T(t)\ dt\\
y^{(n-2)}(t)&=y(t_0)+y'(t_0)+\int_{t_0}^{t}\int_{t_0}^{t}B^H\ T(t)\ dt\\
&\vdots\\
y'(t)&=\sum_{i=0}^{n-1}y^{(i)}(t_0)+\int_{t_0}^{t}\int_{t_0}^{t}\ldots\int_{t_0}^{t}B^H\ T(t)\ dt\\
y(t)&=\sum_{i=0}^{n}y^{(i)}(t_0)+\int_{t_0}^{t}\int_{t_0}^{t}\ldots\int_{t_0}^{t}B^H\ T(t)\ dt
\end{aligned} \quad (11)$$

Substituting Eq. 11 in Eq. 8 and collocating at the collocation points,

$$x_i=\cos\left[\frac{(2i+1)\pi}{4n}\right],$$

where i=0,1,…,n and n is any positive integer, we obtain a system of linear or nonlinear equations depending on the linearity or non linearity of the given equation Eq.1. Solving these system of equations we obtain the Chebyshev polynomial coefficients B^H which yields the solution.

IV. NUMERICAL RESULTS AND DISCUSSION

In order to illustrate the performance of our method, some numerical examples are taken and solved by the method of study. In these numerical computation each graph shows the numerical error of our approximate solution.Here we use the symbol ϕ_A for approximate solution and ϕ_E for exact solution.

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Example 4.1 We consider the integro-differential equation:

$$\phi'(x) + \int_{-1}^{1} \frac{\phi(t)}{t-x} dt = f(x)$$

where $f(x) = \frac{2}{3}x^2 + 7x^4 + x^5 \ln(\frac{1-x}{1+x}) + \frac{2}{5}$

The exact solution is $\phi_E(t) = t^5$

Solution: The given integro-differential equation can be converted into the following ordinary differential equation using Eq. 6 and obtain

$$\ln\left(\frac{1-x}{1+x}\right)\phi_A(x) + 3\phi_A'(x) + \sum_{k=2}^{9} a_k(x)\phi_A^k(x) = f(x) \qquad (12)$$

where $a_2(x), a_3(x) \dots a_9(x)$ are calculated using Eq.7

Let us assume, the solution in terms of chebyshev polynomial as

$$\phi_A^{(9)}(t) = \sum_{i=0}^{9} C_i T_i(t)$$

this gives

$$\phi_A^{(9)}(t) = c_2(2t^2-1) + c_3 t(4t^2-3) + c_4(8t^4-8t^2+1) + c_5 t(16t^4-20t^2+5) + c_6(32t^6-48t^4+18t^2-1)$$
$$+c_7 t(64t^6-112t^4+56t^2-7) + c_8(128t^8-256t^6+160t^4-32t^2+1) + c_9 t(256t^8-576t^6+432t^4-120t^2+9)$$
$$+c_1 t + c_0 \qquad (13)$$

Substitution of Eq.13 and the expression obtained after successive integration in the Eq.12,yield

$$\sum_{i=2}^{9} a_i(t)\phi_i(t) + 3\phi_1(t) + \phi_0(t)\log\left(\frac{1-t}{t+1}\right) = f(t) \qquad (14)$$

Now collocating the Eq. 14 using the collocation points

$$x_i = \cos\left[\frac{(2i+1)\pi}{4n}\right] \text{ where } i = 0,1,\dots 9 \text{ and } n = 9$$

we get a system of equation, solving which we get

$$\phi_A(t) = t^5 - 1.13359t^9\times10^{-14} + 1.2336t^{10}\times10^{-14} + 1.33974t^{11}\times10^{-14} - 6.23522t^{12}\times10^{-14} + 9.95337t^{13}\times10^{-14}$$
$$-9.4424t^{14}\times10^{-14} + 5.76766t^{15}\times10^{-14} - 2.24047t^{16}\times10^{-14} + 5.0674t^{17}\times10^{-15} - 5.10334t^{18}\times10^{-16}$$

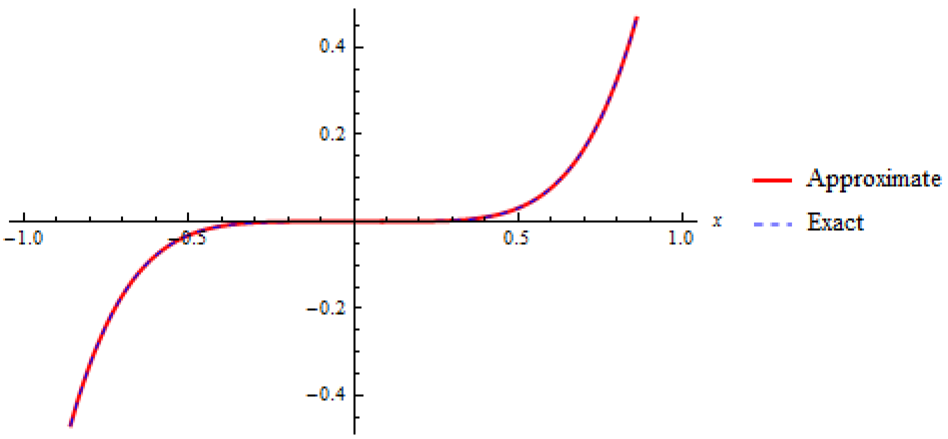

Figure 1: Comparison of $\phi_A(t)$ and $\phi_E(t)$ for Example:3.1

Example 4.2 Consider the integro-differential equation of the form

$$\ln(\frac{1-x}{1+x}) + x^2 \int_{-1}^{1} \frac{\phi(t)}{t-x} dt = f(x)$$

where $f(x) = \frac{2}{9}x^2 + \frac{2}{7}x^4 + \frac{2}{5}x^6 + \frac{2}{3}x^8 + 2x^{10} + (9x^8 + x^{11})\ln(\frac{1-x}{1+x})$

The exact solution for this equation is $\phi_E(t) = t^9$

Solution Converting the integro-differential equation in example(3.2) into an ordinary differential equation we get

$$\ln\left(\frac{1-x}{1+x}\right)\ln\left(\frac{1-x}{1+x}\right)\phi_A(x) + (\ln\left(\frac{1-x}{1+x}\right) + 2x^2)\phi_A'(x) + x^2\left(\sum_{k=2}^{9} a_k(x)\phi_A^k(x)\right) = f(x) \qquad (15)$$

After the computation the approximate solution is

$\phi_A(t) = -0.000319145t^{18} + 0.00301595t^{17} - 0.012737t^{16} + 0.0314615t^{15} - 0.0496434t^{14} + 0.0504504t^{13} - 0.0305029t^{12} + 0.00615733t^{11} + 0.00607962t^{10} + 0.994615t^9$

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(16)

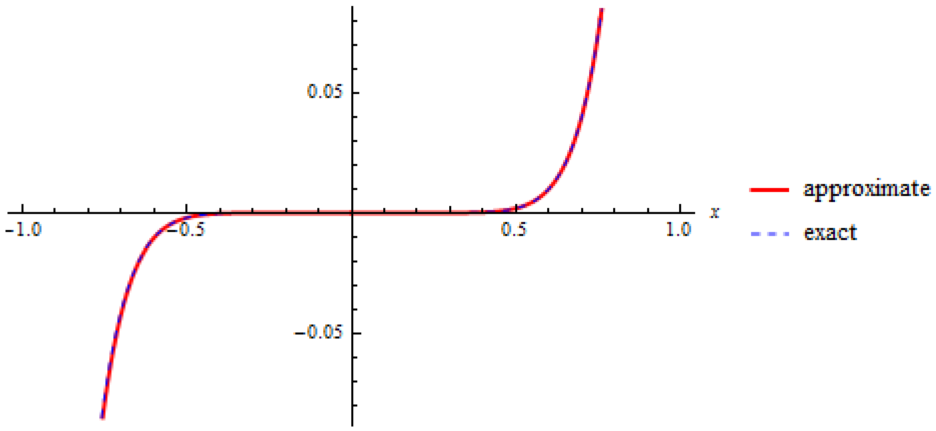

Figure 2: Comparison of $\phi_A(t)$ and $\phi_E(t)$ for Example:3.2

Example 4.3 Consider the integro-differential equation of the form

$$(1+x^2)^2\phi'(x) + x^3\int_{-1}^{1}\frac{\phi(y)}{y-x}dy = 1 - x^2 + x^3\left(\frac{\pi+\ln\left(\frac{1-x}{1+x}\right)}{2x^2+2}\right)$$

and the exact solution is $\phi_E(y) = \frac{y}{y^2+1}$

Solution: The given equation is converted into ordinary differential equation as

$$(1+x^2)^2\ln\left(\frac{1-x}{1+x}\right)\phi_A(x) + ((1+x^2)^2 + 2x^3)\phi_A'(x) + x^3\left(\sum_{k=2}^{9} a_k(x)\phi_A^k(x)\right) = f(x) \qquad (17)$$

As proceeding like example 3.1,we get the approximate solution as

$$\phi_A(t) = 0.00179641t^{18} - 0.0168997t^{17} + 0.0709358t^{16} - 0.173742t^{15} + 0.271579t^{14}$$
$$-0.276399t^{13} + 0.171282t^{12} - 0.0353256t^{11} - 0.0339583t^{10} + 0.584695t^9 - t^7 + t^5 - t^3 + t \qquad (18)$$

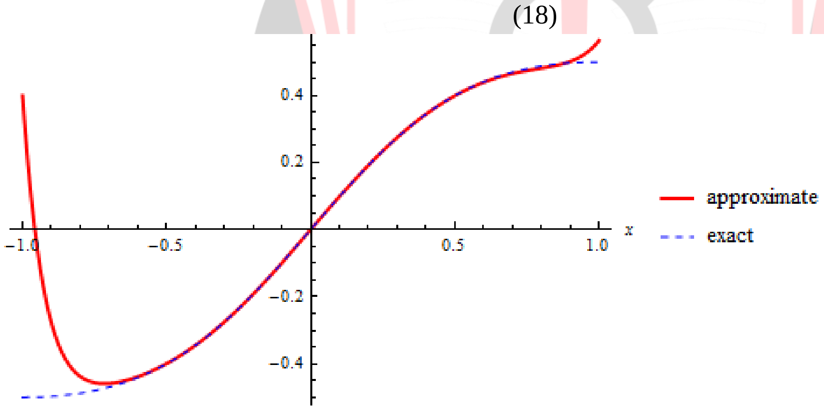

Figure 3: Comparison of $\phi_A(t)$ and $\phi_E(t)$ for Example:3.3

Example 4.4 Consider the integro-differential equation of the form

$$\phi'(x) + \frac{1}{\pi^5}\int_{-1}^{1}\frac{\phi(t)}{t-x}dt = \frac{1}{\pi^4}x^2 + 2\left(\frac{1}{\pi^5+1}\right)x - \frac{1}{\pi^4}$$

The exact solution is $\phi_E(t) = t^2 - 1$

Solution: The given integro-differential equation is converted into ordinary differential equation as

$$\ln\left(\frac{1-x}{1+x}\right)\phi_A(x) + \left(1+\frac{2}{\pi^5}\right)\phi_A'(x) + \frac{1}{\pi^5}\left(\sum_{k=2}^{9} a_k(x)\phi_A^k(x)\right) = f(x) \qquad (19)$$

after simplification we get the approximate solution as

$$\phi_A(t)=0.0000197327t^{18}-0.0029809t^{17}+0.00272506t^{16}+0.0585527t^{15}+0.0664694t^{14}$$
$$-0.256788t^{13}-0.756664t^{12}-0.748276t^{11}-0.198056t^{10}+0.23892t^9+0.437917t^8+0.548819t^7$$
$$+0.414896t^6+0.141626t^5+0.0227606t^4+0.0148971t^3+0.997129t^2-0.0111463t-1.00357 \qquad (20)$$

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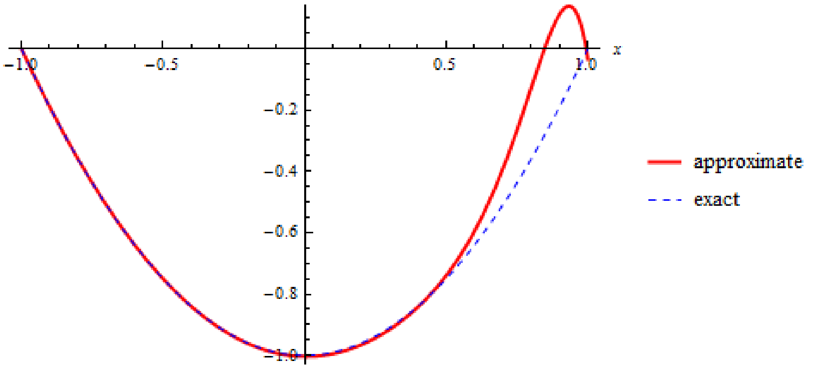

Figure 4: Comparison of $\phi_A(t)$ and $\phi_E(t)$ for Example:3.4

Example 4.5 Consider the integro-differential equation of the form

$$\phi' + \int_{-1}^{1} \frac{\phi(t)}{t-x} dt = f(x)$$

where $f(x) = x^3\pi - 5x^2 - x\pi + \frac{7}{3} + (x^3 - x)\ln(x+1) - (x^3 - x)\ln(x-1)$

the exact solution is $\phi_E(t) = -t^3 + t$

Solution: The integro-differential equation in example 3.4 may be converted into following ordinary differential equation

$$\ln\left(\frac{1-x}{1+x}\right)\phi_A(x) + (1 + 2\frac{1}{\pi^5})\phi_A'(x) + \sum_{k=2}^{9} a_k(x)\phi_A^k(x) = f(x) \quad (21)$$

After computation the approximate solution is

$$\phi_A(t) = -0.00235331t^{20} + 0.0234908t^{19} - 0.103184t^{18} + 0.257722t^{17} - 0.391416t^{16}$$
$$+0.359347t^{15} - 0.241429t^{14} + 0.425653t^{13} - 1.19153t^{12} - 0.338085t^{11} - t^3 + t \quad (22)$$

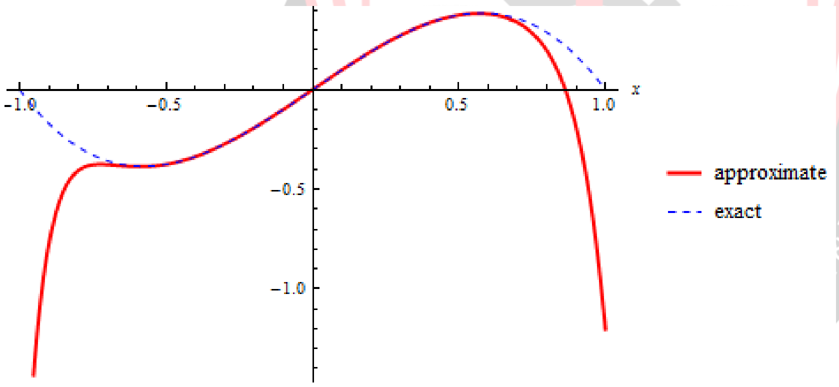

Figure 5: Comparison of $\phi_A(t)$ and $\phi_E(t)$ for Example:3.5

V. CONCLUSION

In this manuscript, we have applied Chebyshev polynomials collocation method to solve integro-differential equations with Cauchy's kernel and inhomoheneous source terms. On comparison of the solution obtained by the present method with exact solution, we observe that in the prescribed domain it's accuracy level is quite convincing. The comparisons are shown in Fig.1 to Fig.5.

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