Ideals in Ternary-Semirings

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Abstract: **Here, we introduce the notions of left noetherian, strict ternary semiring, t-austere, pseudo complement, dense, ternary semi-integral (integral, resp.) domain, Gel'fand ternary semiring. It is proved that In a ternary semiring M the conditions (1) M is left noetherian, (2) Any nonempty collection of ideals of M has maximal element, (3) Every t-ideal of M is finitely generated are equivalent. If M be a ternary multiplicative cancellative ternary semiring, then $\{0\} \cup [M \setminus K^{+}(M)] \in$ t-ideal(M). Let M is a left t-austere then (i)M is entire, (ii) M is zero sum free or ternary ring, (iii) If M is +vely cancellative then it is ternary multiplicatively cancellative as well. A t-austere commutative ternary semiring which is not zero sum free is a ternary field. Further it is proved that if M is a commutative ternary semiring then the set Q of all elements P of t-ideals ideal(M) such that $p \in P \Rightarrow p \lhd q$ for some $q \in P$ is a ternary subsemiring of ideal(M).**

Keywords — t-ideal, left noetherian, strict ternary semiring, t-austere, pseudo complement, dense, ternary semi-integral (integral, resp.) domain, Gel'fand ternary semiring.

I. INTRODUCTION

Ideals play a basic role in ring theory, semigroup theory and many algebraic structures and hence ideal role is no less importance, therefore, we consider special type of ideals in ternary semirings. For the definition of ternary semiring refer to the references. Throughout this paper M denotes the ternary semiring unless otherwise stated.

II. MAIN RESULTS

Definition 2.1: A subset $P \neq \emptyset$ of M is said to be ***left*(*lateral, right*) *t-ideal*** of M if

(1) $q, m \in P \Rightarrow q + m \in P$.

(2) $m, c \in M, q \in P \Rightarrow mcq\ (mqc, qmc) \in P$.

Definition 2.2: A subset $P \neq \emptyset$ of M is said to be ***two sided t-ideal*** of M if P is both left as well as right *t*-ideal of M.

Definition 2.3: A subset $P \neq \emptyset$ of M is said to be ***t-ideal*** of M if P is left, lateral as well as right *t*-ideal of M.

Example 2.4: Let *N* be the set of natural numbers Then N is a ternary semiring and A = 3N is a t-ideal of the ternary semiring *N*.

Definition 2.5: A sub set $P \neq \emptyset$ of M is known as ***subtractive*** iff $f \in P$ and $f + l \in P \Rightarrow l \in P$. P is known strong if $f + l \in P \Rightarrow f \in P \;\&\; l \in P$.

Definition 2.6: A sub set $P \neq \emptyset$ of M known as ***semi-subtractive*** iff $f \in P \cap V(M) \Rightarrow -f \in P \cap A(M)$, where A(M) is the set of all additive inverses of M.

Example 2.7: The t-ideal $P = N \setminus \{1\}$ of N is semi-subtractive.

Definition 2.8: M known as ***left noetherian*** iff the ascending chain condition holds on left ideals.

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Theorem 2.9: In a ternary semiring M the conditions
(1) M is left noetherian,
(2) Any nonempty collection of ideals of M has maximal element,
(3) Every t-ideal of M is finitely generated are equivalent.

***Proof*: (1) ⇒ (2):** Suppose $\mathfrak{C}$ is the finite set of left t-ideals of M and $P_1 \in \mathfrak{C}$. If P_1 is not any element of $\mathfrak{C}$ then the statement is true. If P_1 contained in P_2 of $\mathfrak{C}$ then If P_2 is not in $\mathfrak{C}$ then the statement is true. If P_2 contained in P_3 of $\mathfrak{C}$ then If P_3 is not in $\mathfrak{C}$ then the statement is true. Proceeding like that then by (1) the method must terminate after a finite steps and hence $\mathfrak{C}$ has a maximal element.
(2) ⇒ (3): Suppose P is a left t-ideal of M & $\mathfrak{C}$ is a set of finitely generated t-ideals of M contained in P. Since the set is non empty and hence by condition (2) it has maximal element I = MM$\{p_1, p_2, \ldots\ldots p_m\}$. For each $q \in$ P, suppose, I_q = MM$\{p_1, p_2, \ldots\ldots p_m, q\}$ but for maximal condition $I = I_q$. Therefore, in particular, $p \in I\ \forall\ q \in P$. Thus P = I, therefore P is finitely generated.
(3) ⇒ (1): Let $P_1 \subseteq P_2 \subseteq P_3 \subseteq \ldots\ldots\ldots$is an ascending chain of left t-ideals of M as well as $P = \bigcup_{i=1}^{\infty} P_i$. Therefore, P is a left t-ideal of M. By condition (3), P is finitely generated, say P = MM$\{p_1, p_2, \ldots\ldots p_m\}$, that is ∃ an index $t \ni P \subseteq P_t \subseteq P$. Thus $P_i = P_t\ \forall\ i \geq t$.

Definition 2.10: A ternary semiring M is said to be ***multiplicatively left*(lateral, right) *cancellative*** (MLC, MLLC, MRC) if $pbx = pby (pxb = pyb, xpb = ypb) \Rightarrow x = y$ $\forall\ p, b, x, y \in$ M. A ternary semiring M is said to be ***multiplicatively cancellative*** (MC) if it is MLC, MRC & MLLC. The set of all additively cancellative elements are denoted by $K^+(M)$ and ternary multiplicatively cancellative elements of M are denoted by $K^\times(M)$.

Theorem 2.11:If M is multiplicative cancellative , then $\{0\} \cup [M \setminus K^+(M)] \in$ t-*ideal*(M).

***Proof*:** Construct P = $\{0\} \cup [M \setminus K^+(M)] \in$ t-ideal(M). If $0 \neq p, q \in P$ & $p + q \in K^+(M)$, then $p + s = p + r \Rightarrow p + q + s = p + q + r \Rightarrow s = r$ and hence $p \in K^+(M)$. This is a contradiction. Thus sum of elements of P again in P. If $0 \neq p \in P$ & $0 \neq s, r \in M$ such that $srp \in K^+(M)$, then $p + t = p + u \Rightarrow srp + srt = srp + sru \Rightarrow srt = sru \Rightarrow t = u$. This is a contradiction and hence $srp \in P\ \forall\ s, r \in M$. Similarly, spr, $psr \in P\ \ \forall\ s, r \in M$. If $1 \in P \Rightarrow P = M$. Otherwise, P is a t-ideal of M in all the cases, P ∈ t-*ideal (M).*

Definition 2.12: M is known as a ***strict ternary semiring*** or ***zero sum free*** provided $d + f = 0$ implies $d = 0$ & $f = 0$.

Definition 2.13: An element u of a ternary semiring M is said to be an ***absorbing*** of T provided $u + n = n = n + u$ & $uab = aub = abu = u\ \forall\ a, b, n \in$ M.

Note 2.14: In rings every zero is absorbing zero, but in ternary semiring not every zero is absorbing zero, to show that consider the following.

Example 2.15: Consider the set of positive integers Z^+ with the operations $n + f = lcm(n,f)$ & ternary multiplication. Then Z^+ is a ternary semiring with zero element 1, but 1 is not an absorbing zero since $1.1.a = a.1.1 = a \neq 1$ for any $a \in Z^+$ and $a \neq 1$.

Definition 2.16: M is said to be ***zero divisor free*** (ZDF) if for $s, l, p \in M$, $[slp] = 0$ implies that $s = 0$ or $l = 0$ or $p = 0$. M is ***entire*** if M has no zero divisors.

Definition 2.17: A ternary semiring M which has left t-ideals which is non zero subtractive is known as ***left t-austere***. In the same manner, we can define ***lateral t-austere*** as well as ***right t-austere***.

Example 2.18: Let $u \notin$ M & E = M ∪ $\{u\}$. The operations defined as if $m, s, p \in M \Rightarrow m + s$ and $msp \in$ M and $l, q \in E \ni l + u = u + l = l$ & $lqu = qul = ulq = u$. Then E is a ternary semiring with +ve identity u, which is both zero sum free and entire. Since this ternary semiring is t-austere.
Definition 2.19: If s and t are elements of a frame [18] $(L, \vee, \wedge)$ then the ***pseudo complement*** of b relative to a , denoted $(s : t)$, is the unique largest element u of L satisfying $t \wedge u \leq s$. T he pseudo complement of an element s of L is $(0 : s)$. If $(0 : s) =0$, then s is ***dense*** in L. The dense elements of a frame are precisely those elements which are not zero divisors.

Theorem 2.20: Let M is a left t-austere then

(i) M is entire
(ii) M is zero sum free or ternary ring
(iii) **If M is +vely cancellative then it is ternary MC as well.**

***Proof* : (i)** Let M is a non zero subtractive left t-ideals and s, v, a are non zero elements of M such that $sva = 0$, then $0 \neq s \in (0, v)$ & $0 \neq s \in (0, a)$ and hence $(0, v) = (0, a) = M$ since otherwise $(0, v)$, $(0, a)$ are non zero subtractive left t-ideals of M. But this is not possible because $1 \notin (0, v)$. Therefore M should be entire.

(ii) Let M is not zero sum free $V(M) \neq \{0\}$. Here V(M) is clearly subtractive left t-ideal of M. i.e., V(M) = M. Therefore M is a ternary ring.

(iii) Suppose M is +vely cancellative then for $s, t, u, w \in M \ni P = \{ x \in M / xst = xuw\}$is a subtractive left t-ideal of M or equal to M itself. Here, $P \neq \{0\}$ then P = M and so 1 ∈

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P. i.e., $s = u$ or $t = w$, therefore M is a left ternary multiplicative cancellative.

Definition 2.21: A commutative ternary semi ring(ring) is called a ***ternary semi-integral*** (***integral***, resp.) ***domain*** if it is zero divisor free.

Theorem 2.22: A t-austere commutative ternary semiring which is not zero sum free is a ternary field.

***Proof*:** If M is a t-austere commutative ternary semiring which is not zero sum free, so by theorem 2.20, M is ternary semi-integral domain, if $0 \neq m, n \in$ M then Mmn is a subtractive t-ideal of M & M$mn \neq \{0\}$ and so all of M. therefore, $\exists\ l \in$ M $\ni lmn = 1$ and hence M is a ternary field.

Theorem 2.23:If M is a commutative then the set Q of all elements P of t-ideals *ideal*(M) $\ni p \in$ P $\Rightarrow p \lhd q$ for some $q \in$ P is a ternary subsemiring of *ideal*(M).

***Proof*:** As mentioned in [20] note that $0 \lhd 0$ & $p \lhd 1$for all $p \in$ P. thus {0} & M $\in$ Q. Assume that P, H, G $\in$ Q & $p \in$ P, $q \in$ H & $r \in G$ then $\exists\ p' \in$ P, $q' \in$ H & $r' \in$ G such that $p \lhd p'$, $q \lhd q'$ & $r \lhd r'$, then $\exists\ s, t, u \in$ M $\ni pps = qqt = rru = 0$ & $s + p' = t + q' = u + r' = 1$. Thus $v = p'q'r' + p'q's + q'r't + p'r'u \in$ P + H + G $\Rightarrow v + stu = 1$ while $(p + q + r)stu = 0$. Hence $(p + q + r) \lhd v$. Therefore, P + H + G $\in$ Q.

Similarly, if P, H, G $\in$ Q & $p \in$ PHG then $p \in$ P $\cap$ H $\cap$ G $\exists\ q \in$ P, $r \in$ H, $s \in$ G $\ni p \lhd q$, $p \lhd r$, $p \lhd s$ then $\exists\ s, t, u \in$ M $\ni pps = qqt = rru = 0$ & $s + p' = t + q' = u + r' = 1$. Then $stu \in$ P and if $v = p'q'r' + p'q's + q'r't + p'r'u$, we have $v + stu = 1$while $pv1 = 0 \Rightarrow p \lhd stu$. Therefore PHG $\in$ Q. Therefore, P is a ternary subsemiring of *ideal*(M).

Theorem 2.24: If S, A, T are respectively left, lateral and right t-ideals of M then S + A + T is the unique minimal member of the family of all left, lateral, right t-ideals of M containing S, A, T and S ∩ A ∩ T is the unique maximal member of the family of all left, lateral, right t-ideals of M containing S, A, T.

***Proof*:** Obviously, S + A + T contains S, A, T, Conversely, If K is a t-ideal containing S, A, T then K contains all the elements M of the form $s + m + t$, $s \in$ S, $m \in$A, $t \in$ T and hence K contains S + A + T. Similarly, the proof of other part can prove.

Theorem 2.25:If M is a ternary semiring then a sufficient condition for a lattice *t-lideal*(M)[resp. *t-mideal*(M), *t-right*(M), *t-ideal*(M)] to be modular is that each of its members be subtractive.

***Proof*:** Suppose every element of *t-lideal*(M) is subtractive. Let S, T, U be the *t*-lideals of M such that T ∩ S = T ∩ U while T + S = T + U & S ⊆ U. We have to show that S = U. Let $u \in$ U, we can write $u = v + w$ where $v \in$ T and $w \in$ U. Since $w \in$ U therefore by subtractiveness, $v \in$ T ∩ U = T ∩ S. Therefore $u \in$ S. Establishing the desired equality.

Definition 2.25:A ternary semiring M is said to be ***Gel'fand ternary semiring*** iff M = {$m \in$ M/ 1 + $m \in$ U(M), where U(M) is the set of all unities of M}. A non empty sub set S of a ternary semiring M is called ***strong*** iff $a + b \in$ S $\Rightarrow a \in$ S and $b \in$ S.

Example 2.26: Simple ternary semiring surely Gel'fand ternary semiring.

Theorem 2.27: The ternary semiring M is a Gel'fand ternary semiring iff $p + r + q \in$ U(M) ∀ $q \in$ U(M) & ∀ $p,r \in$ M.

***Proof*:** If $p + r + q \in$ U(M) ∀ $q \in$ U(M) & ∀ $p, r \in$ M. Then surely M is Gel'fand ternary semiring. Conversely, suppose M is Gel'fand ternary semiring, let $p, r \in$ M & $q \in$ U(M). Then $r = q^{-1}pr + 1$ is a unit of M and hence $p + r + q = pqr \in$ U(M).

Theorem 2.28:If M is a Gel'fand ternary semiring then K^+(M) is a strong t-ideal of M.

***Proof*:** we know that K^+(M) is closed under addition. Let $q \in K^+$(M) & let $p, r \in$ M. Suppose $pqr + a = pqr + b \Rightarrow q(1 + p)(1 + r) + a = q(1 + p)(1 + r) + b \Rightarrow q + a(1 + p)^{-1}(1 + r)^{-1} = q + b(1 + p)^{-1}(1 + r)^{-1} \Rightarrow a(1 + p)^{-1}(1 + r)^{-1} = b(1 + p)^{-1}(1 + r)^{-1} \Rightarrow a = b$. Thus $pqr \in K^+$(M). Similarly, prq, $rpq \in K^+$(M). Therefore, K^+(M) is a t-ideal of M. Now suppose that $u + v \in K^+$(M) $\Rightarrow u + a = u + b \Rightarrow u + v + a = u + v + b \Rightarrow a = b$. Thus $u \in K^+$(M) & similarly $v \in K^+$(M) and therefore K^+(M) is strong.

III. Conclusion

In this paper we investigated mainly about different types of t-ideals in ternary semiring and different kinds of ternary semirings. D. Madhusudhana Rao, G. Srinivasa Rao invrstigated and studied about ideals in ternary semirings. After that D. Madhusudhana Rao and P. Sivaprasad investigated and develop the literature of partially ordered ternary semirings.

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