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Unitarily Congruence of Conjugate Unitary Matrices

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Abstract:. Concept of conjugate unitary matrices are given. Unitarily congruent between two complex $n \times n$ matrices, extended to transpose, secondary transpose of conjugate unitary matrix are introduced. Unitarily congruence of conjugate unitary matrices are also introduced and some results are derived

Keywords: unitary matrix, transpose of a matrix, secondary transpose of a matrix, secondary conjugate transpose of a matrix, conjugate unitary matrix.

Mathematics Subject Classification: 15B10

I. INTRODUCTION

Anna Lee [1] has shown that for a complex matrix A, the usual transpose A^T and secondary transpose A^s are related as $A^s = VA^TV$, where 'V' is the permutation matrix with units in its secondary diagonal. Also $\bar{A}^s$ denotes the conjugate secondary transpose of A. i.e., $\bar{A}^s = (c_{ij})$ where $c_{ij} = \overline{a_{n-j+1,n-i+1}}$ [2].Matrix diagonalization is a wide range of application in the matrix theory has been discussed and problems of simultaneous congruence diagonalization of two matrices, and gave some theorems and proofs in [6]. A matrix $A \in C_{nxn}$ is unitarily congruent to $\bar{A}$ via a symmetric unitary matrix was discussed in [7].Unitary congruence between two matrices was disussed in[8]. In this paper our intension is to prove some theorems on unitarily congruence of conjugate unitary matrices.

1.1 NOTATIONS : Let C_{nxn} be the space of nxn complex matrices of order n. For $A \in C_{nxn}$. Let A^T,$\bar{A}$, A^*, A^s denote transpose, conjugate, conjugate transpose, secondary transpose, of a matrix A respectively.

A matrix $A \in C_{nxn}$ is called normal if $AA^* = A^*A$ [5]

A matrix $A \in C_{nxn}$ is called unitary if $AA^* = A^*A = I$ [9]

A matrix $A \in C_{nxn}$ is called conjugate normal if $AA^* = \overline{A^*A}$ [3]

A matrix $A \in C_{nxn}$ is called is called conjugate unitary if $AA^* = \overline{A^*A} = I$ [4]

II. UNITARILY CONGRUENCE OF CONJUGATE UNITARY MATRICES

In this section our intension is to prove some theorems on unitarily congruence of conjugate unitary matrices.

Definition 2.1: Two matrices $A, B \in C_{n\times n}$ are unitarily congruent if there is a unitary

matrix U of same size such that $A = UBU^T$.

Theorem 2.2: Let $A, B \in C_{n\times n}$ are conjugate unitary matrices. If B is unitarily congruent to A then the secondary transpose of B is conjugate unitary matrix.

Proof: Given $A, B \in C_{n\times n}$ are conjugate unitary matrices.

$\Longrightarrow AA^* = \overline{A^*A} = I$ and $BB^* = \overline{B^*B} = I$

Assume that B is unitarily congruent to A.

i.e., $B = U^TAU$

To show $B^s(B^s)^* = \overline{(B^s)^*B^s} = I$

Case (i) $BB^* = (U^TAU)(U^TAU)^*$

$= U^TAUU^*A^*(U^T)^*$

$= U^TAUU^*A^*(U^*)^T$

$= U^TAA^*(U^*)^T \quad (\because \; UU^* = I)$

Taking secondary transpose on both sides, we have,

$$(BB^*)^s = (U^TAA^*(U^*)^T)^s$$

$$\Longrightarrow (B^*)^sB^s = ((U^*)^T)^s(A^*)^sA^s(U^T)^s$$

$$\Longrightarrow (B^*)^sB^s = ((U^*)^T)^s(AA^*)^s(U^T)^s$$

$$\Longrightarrow (B^*)^sB^s = ((U^*)^T)^sI^s(U^T)^s$$

$$\Longrightarrow (B^*)^sB^s = ((U^*)^T)^s(U^T)^s \quad (\because \; I^s = I)$$

$$\Longrightarrow (B^*)^sB^s = (U^T(U^*)^T)^s$$

$$\Longrightarrow (B^*)^sB^s = ((U^*U)^T)^s$$

$$\Longrightarrow (B^*)^sB^s = (I^T)^s \quad (\because \; U^*U = I)$$

$$\Longrightarrow (B^*)^sB^s = I^s \quad (\because \; I^T = I)$$

$$\Longrightarrow (B^s)^*B^s = I \quad (\because \; I^s = I)$$

Taking conjugate on both sides, we have,

$$\overline{(B^s)^*B^s} = \bar{I}$$

$$\overline{(B^s)^*B^s} = I$$

Case(ii) $B^*B = (U^TAU)^*(U^TAU)$

$= U^*A^*(U^T)^*U^TAU$

$= U^*A^*(UU^*)^TAU$

$= U^*A^*I^TAU \quad (\because \; UU^* = I)$

$= U^*A^*IAU \quad (\because \; I^T = I)$

$= U^*A^*AU$

Taking secondary transpose on both sides, we have,

$$(B^*B)^s = (U^*A^*A\,U)^s$$

$$\Longrightarrow B^s(B^*)^s = U^sA^s(A^*)^s(U^*)^s$$

$$\Longrightarrow B^s(B^s)^* = U^s(A^*A)^s(U^*)^s$$

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Taking conjugate on both sides, we have,

$$\overline{B^s(B^s)^*} = \overline{U^s}\,\overline{(A^*A)^s}\,\overline{(U^*)^s}$$

$$\Longrightarrow \overline{B^s(B^s)^*} = \overline{U^s}\,\overline{\overline{(U^*)^s}} \qquad (\because \overline{A^*A} = I)$$

$$\Longrightarrow \overline{B^s(B^s)^*} = \overline{(U^*U)^s}$$

Again taking conjugate on both sides, we have,

$$\Longrightarrow B^s(B^s)^* = (U^*U)^s$$

$$\Longrightarrow B^s(B^s)^* = I^s \qquad (\because U^*U = I)$$

$$\Longrightarrow B^s(B^s)^* = I \qquad (\because I^s = I)$$

Therefore, in both cases, we have,

$$B^s(B^s)^* = \overline{(B^s)^*B^s} = I$$

$\Longrightarrow B^s$ is conjugate unitary matrix.

i.e., The secondary transpose of B is conjugate unitary.

Theorem 2.3: Let $A, B \in C_{n\times n}$ are conjugate unitary matrices. If A is unitarily congruent to B then AA^* is conjugate unitary matrix.

Proof: Given $A, B \in C_{n\times n}$ are conjugate unitary matrices.

$$\Longrightarrow AA^* = \overline{A^*A} = I \text{ and } BB^* = \overline{B^*B} = I$$

Assume that A is unitarily congruent to B.

i.e., $A = UBU^T$

$$\Longrightarrow A^* = (UBU^T)^*$$

$$= (U^T)^*B^*U^*$$

To show AA^* is conjugate unitary matrix

That is to show $(AA^*)\,(AA^*)^* = \overline{(AA^*)^*(AA^*)} = I$

Case(i) (AA^*)

$$(AA^*)^* = \big((UBU^T)(UBU^T)^*\big)\big((UBU^T)(UBU^T)^*\big)^*$$

$$= UBU^T(U^T)^*B^*U^*\,UBU^T(U^T)^*B^*U^*$$

$$= UBU^T(U^T)^*B^*BU^T(U^T)^*B^*U^* \qquad (\because U^*U = I)$$

$$= UBU^T(U^*)^TB^*BU^T(U^*)^TB^*U^*$$

$$= UB(U^*U)^TB^*B(U^*U)^TB^*U^*$$

$$= UB(I)^TB^*B(I)^TB^*U^* \qquad (\because U^*U = I)$$

$$= UBIB^*BIB^*U^* \qquad (\because I^T = I)$$

$$= U(BB^*)(BB^*)U^*$$

$$= UU^* \qquad (\because BB^* = I)$$

$$= I \qquad (\because UU^* = I)$$

Case(ii)

$$(AA^*)^*(AA^*) = \big((UBU^T)(UBU^T)^*\big)^*\big((UBU^T)(UBU^T)^*\big)$$

$$= UBU^T(U^T)^*B^*U^*UBU^T(U^T)^*B^*U^*$$

$$= UBU^T(U^T)^*B^*BU^T(U^T)^*B^*U^* \qquad (\because U^*U = I)$$

$$= UBU^T(U^*)^TB^*BU^T(U^*)^TB^*U^*$$

$$= UB(U^*U)^TB^*B(U^*U)^TB^*U^*$$

$$= UB(I)^TB^*B(I)^TB^*U^* \qquad (\because U^*U = I)$$

$$= UBIB^*BIB^*U^* \qquad (\because I^T = I)$$

$$= UBB^*BB^*U^*$$

Taking conjugate on both sides, we have,

$$\overline{(AA^*)^*(AA^*)} = \overline{UB}\;\overline{B^*B}\;\overline{B^*U^*}$$

$$= \overline{UB}\;\overline{B^*U^*} \qquad (\because \overline{B^*B} = I)$$

Again taking conjugate on both sides, we have,

$$(AA^*)^*(AA^*) = UBB^*U^*$$

$$= UU^* \qquad (\because BB^* = I)$$

$$= I \qquad (\because UU^* = I)$$

Taking conjugate on both sides, we have,

$$\overline{(AA^*)^*(AA^*)} = \bar{I}$$

$$= I \qquad (\because \bar{I} = I)$$

Therefore, in both cases, we have,

$$(AA^*)\,(AA^*)^* = \overline{(AA^*)^*(AA^*)} = I$$

$\Longrightarrow AA^*$ is conjugate unitary matrix.

Theorem 2.4: Let $A, B \in C_{n\times n}$ are conjugate unitary matrices. If B is unitarily congruent to A then BB^* is conjugate unitary matrix.

Proof: Given $A, B \in C_{n\times n}$ are conjugate unitary matrices.

$$\Longrightarrow AA^* = \overline{A^*A} = I \text{ and } BB^* = \overline{B^*B} = I$$

Assume that B is unitarily congruent to A.

i.e., $B = U^TAU$

$$\Longrightarrow B^* = (U^TAU)^*$$

$$= U^*B^*(U^T)^*$$

To show BB^* is conjugate unitary matrix

That is to show $(BB^*)\,(BB^*)^* = \overline{(BB^*)^*(BB^*)} = I$

Case(i) (BB^*)

$$(BB^*)^* = \big((U^TAU)(U^TAU)^*\big)\big((U^TAU)(U^TAU)^*\big)^*$$

$$= U^TAUU^*A^*\,(U^T)^*U^TAUU^*A^*(U^T)^*$$

$$= U^TAA^*\,(U^T)^*U^TAA^*(U^T)^* \qquad (\because UU^* = I)$$

$$= U^T\,(U^*)^TU^T(U^T)^* \qquad (\because AA^* = I)$$

$$= U^T\,(UU^*)^T(U^T)^*$$

$$= U^T\,I^T(U^T)^* \qquad (\because UU^* = I)$$

$$= U^T\,I(U^*)^T \qquad (\because I^T = I)$$

$$= (U^*U)^T$$

$$= I^T \qquad (\because U^*U = I)$$

$$= I \qquad (\because I^T = I)$$

Case(ii)

$$(BB^*)^*(BB^*) = \big((U^TAU)(U^TAU)^*\big)^*\big((U^TAU)(U^TAU)^*\big)$$

$$= U^TAUU^*A^*(U^T)^*U^TAUU^*A^*(U^T)^*$$

$$= U^TAA^*(U^*)^TU^TAA^*(U^*)^T \qquad (\because UU^* = I)$$

$$= U^TAA^*(UU^*)^TAA^*(U^*)^T$$

$$= U^TAA^*I^TAA^*(U^*)^T \qquad (\because UU^* = I)$$

$$= U^TAA^*IAA^*(U^*)^T \qquad (\because I^T = I)$$

$$= U^T(U^*)^T \qquad (\because AA^* = I)$$

$$= (U^*U)^T$$

$$= I^T \qquad (\because U^*U = I)$$

$$= I \qquad (\because I^T = I)$$

Taking conjugate on both sides, we have,

$$\overline{(BB^*)^*(BB^*)} = \bar{I}$$

$$\overline{(BB^*)^*(BB^*)} = I \qquad (\because \bar{I} = I)$$

Therefore, in both cases, we have,

$$(BB^*)\,(BB^*)^* = \overline{(BB^*)^*(BB^*)} = I$$

$\Longrightarrow BB^*$ is conjugate unitary matrix.

Theorem 2.5: Let $A, B \in C_{n\times n}$ are conjugate unitary matrices. If B is unitarily congruent to A then the transpose of B is conjugate unitary matrix.

Proof: Given $A, B \in C_{n\times n}$ are conjugate unitary matrices.

$$\Longrightarrow AA^* = \overline{A^*A} = I \text{ and } BB^* = \overline{B^*B} = I$$

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Assume that B is unitarily congruent to A.

i.e., $B = U^T AU$

$\Longrightarrow B^T = (U^T AU)^T$

To show the transpose of B is conjugate unitary matrix.

That is to show $B^T(B^T)^* = \overline{(B^T)^* B^T} = I$

Case(i) $B^T(B^T)^* = (U^T AU)^T((U^T AU)^T)^*$

$= U^T A^T U(U^T A^T U)^*$

$= U^T A^T UU^*(A^T)^*(U^T)^*$

$= U^T A^T(A^T)^*(U^T)^*$

$(\because\ UU^* = I\ )$

$= U^T A^T(A^*)^T(U^T)^*$

$= U^T(A^*A)^T(U^T)^*$

Taking conjugate on both sides, we have,

$\overline{B^T(B^T)^*} = \overline{U^T(A^*A)^T(U^T)^*}$

$\overline{B^T(B^T)^*} = \overline{U^T I^T(U^T)^*}$ $\quad (\because \overline{A^*A} = I\ )$

$\overline{B^T(B^T)^*} = \overline{U^T I(U^T)^*}$ $\quad (\because\ I^T = I\ )$

Again taking conjugate on both sides, we have,

$B^T(B^T)^* = U^T(U^T)^*$

$B^T(B^T)^* = U^T(U^*)^T$

$B^T(B^T)^* = (U^*U)^T$

$B^T(B^T)^* = I^T$

$(\because\ U^*U = I\ )$

$B^T(B^T)^* = I$ ………………………………………….. **(2.5.1)**

Case(ii) $(B^T)^*B^T = ((U^T AU)^T)^*(U^T AU)^T$

$= (U^T A^T U)^* U^T A^T U$

$= U^*(A^T)^*(U^T)^* U^T A^T U$

$= U^*(A^T)^*(U^*)^T U^T A^T U$

$= U^*(A^T)^*(UU^*)^T A^T U$

$= U^*(A^T)^* I^T A^T U$

$(\because\ UU^* = I\ )$

$= U^*(A^T)^* I A^T U$

$(\because\ I^T = I\ )$

$= U^*(A^*)^T A^T U$

$= U^*(AA^*)^T U$

$= U^* I^T U$

$(\because\ AA^* = I\ )$

$= U^*U$

$(\because\ I^T = I\ )$

$= I$

$(\because\ U^*U = I\ )$

Taking conjugate on both sides, we have,

$\overline{(B^T)^*B^T} = \bar{I}$

$\overline{(B^T)^*B^T} = I$ $\quad (\because \bar{I} = I)$

……………………….. **(2.5.2)**

Therefore, from equations **(2.5.1)** and **(2.5.2)** we have,

$B^T(B^T)^* = \overline{(B^T)^*B^T} = I$

$\Longrightarrow B^T$ is conjugate unitary matrix.

Theorem 2.6: Let $A, B \in C_{n\times n}$, B is hermitian and A is conjugate unitary matrix. If

AV is unitarily congruent to BV, where V is the permutation matrix

then B is involutary.

Proof: Given (i) $A, B \in C_{n\times n}$

(ii) B is hermitian

(iii) A is conjugate unitary matrix

Assume that AV is unitarily congruent to BV

i.e., $AV = U(BV)U^T$

To show $B^2 = I$

Case(i) $(AV)(AV)^* = (U(BV)U^T)(U(BV)U^T)^*$

$= (U(BV)U^T)(U(BV)U^T)^*$

$= UBVU^T(U^T)^*V^*B^*U^*$

$= UBVU^T(U^*)^T V^*B^*U^*$

$= UBV(U^*U)^T V^*B^*U^*$

$= UBVI^T V^*B^*U^*$ $\quad (\because\ U^*U = I\ )$

$= UBVV^*B^*U^*$ $\quad (\because\ I^T = I\ )$

$= UBVVB^*U^*$ $\quad (\because\ V^* = V\ )$

$= UBV^2B^*U^*$

$= UBIB^*U^*$ $\quad (\because\ V^2 = I\ )$

$= UBB^*U^*$

$AV\,V^*A^* = UBBU^*$ $\quad (\because\ B^* = B\ )$

$AV\,VA^* = UBBU^*$ $\quad (\because\ V^* = V\ )$

$AV^2\,A^* = UBBU^*$

$AI\,A^* = UBBU^*$ $\quad (\because\ V^2 = I\ )$

$I = UBBU^*$ $\quad (\because\ AA^* = I\ )$

Pre multiplying by U^* and Post multiplying by U on both sides, we have,

$U^*U = U^*UBBU^*U$

$I = BB$ $\quad (\because\ U^*U = I\ )$

i.e., $BB = I$

$B^2 = I$

……………………………………….**(2.6.1)**

Case(ii) $(AV)^*(AV) = (U(BV)U^T)^*\,(U(BV)U^T)$

$= (U^T)^*V^*B^*U^*UBVU^T$

$= (U^*)^T V^*B^*U^*UBVU^T$

$= (U^*)^T V^*B^*BVU^T$

$(\because\ U^*U = I\ )$

$V^*A^*AV = (U^*)^T V^*BBVU^T$

$(\because\ B^* = B\ )$

Taking conjugate on both sides, we have,

$\overline{V^*}\ \overline{A^*A}\ \bar{V} = \overline{(U^*)^T V^*BBVU^T}$

$\overline{V^*}\ \bar{V} = \overline{(U^*)^T V^*BBVU^T}$

Again taking conjugate on both sides, we have,

$V^*V = (U^*)^T V^*BBVU^T$

$VV = (U^*)^T V^*BBVU^T$

$(\because\ V^* = V\ )$

$V^2 = (U^*)^T V^*BBVU^T$

$I = (U^*)^T V^*BBVU^T$

Pre multiplying by U^T and Post multiplying by $(U^*)^T$ on both sides, we have,

$U^T(U^*)^T = U^T\,(U^*)^T V^*BBVU^T(U^*)^T$

$(U^*U)^T = (U^*U)^T V^*BBV(U^*U)^T$

$I^T = I^T V^*BBVI^T$ $\quad (\because\ U^*U = I\ )$

$I = V^*BBV$ $\quad (\because\ I^T = I\ )$

$I = VBBV$ $\quad (\because\ V^* = V\ )$

Pre and Post multiplying by V on both sides, we have,

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$$VV = VVBBVV$$
$$V^2 = V^2BBV^2$$
$$I = BB \qquad (\because\ V^2 = I)$$
$$BB = I$$
$$\Longrightarrow\ B^2 = I$$

……………………………………….(2.6.2)

Therefore, from equations **(2.6.1)** and **(1.6.2)** we have,

$$B^2 = I$$

$\Longrightarrow B$ is involuntary

Theorem 2.7: Let $A, B \in C_{n\times n}$, where B is hermitian and A is conjugate unitary matrix. If B is unitarily congruent to A, then B^* is involutary.

Proof: Given (i) $A, B \in C_{n\times n}$

(ii) B is hermitian

(iii) A is conjugate unitary matrix

Assume that B is unitarily congruent to A

i.e., $B = U^TAU$

To show $(B^*)^2 = I$

Case (i) $BB^* = (U^TAU)(U^TAU)^*$

$$= U^TAUU^*A^*(U^T)^*$$
$$= U^TAA^*(U^T)^* \qquad (\because\ UU^* = I)$$
$$= U^T(U^*)^T \qquad (\because\ AA^* = I)$$
$$= (U^*U)^T$$
$$= I^T \qquad (\because\ U^*U = I)$$
$$= I \qquad (\because\ I^T = I)$$
$$B^*B^* = I \qquad (\because\ B = B^*)$$
$$(B^*)^2 = I$$

…………………………………………….(2.7.1)

Case (ii) $B^*B = (U^TAU)^*(U^TAU)$

$$= U^*A^*(U^T)^*U^TAU$$
$$= U^*A^*(U^*)^TU^TAU$$
$$= U^*A^*(UU^*)^TAU$$
$$= U^*A^*I^TAU \qquad (\because\ UU^* = I)$$
$$= U^*A^*AU \qquad (\because\ I^T = I)$$

Taking conjugate on both sides, we have,

$$\overline{B^*B} = \overline{U^*}\,\overline{A^*A}\,\overline{U}$$
$$\overline{B^*B} = \overline{U^*}\ \overline{U} \qquad (\because \overline{A^*A} = I)$$

Again taking conjugate on both sides, we have,

$$B^*B = U^*U$$
$$B^*B = I \qquad (\because\ U^*U = I)$$
$$B^*B^* = I \qquad (\because\ B = B^*)$$
$$(B^*)^2 = I$$

…………………………………..(2.7.2)

Therefore, from equations **(2.7.1)** and **(2.7.2)** we have,

$$(B^*)^2 = I$$

$\Longrightarrow B^*$ is involuntary

III. CONCLUSION

Unitarily congruences of conjugate unitary matrices are obtained. Also unitarily congruence between transpose and secondary transpose of conjugate unitary matrices are derived.

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