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ON CAYLEY-SYMMETRIC Γ- SEMIGROUPS

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ABSTRACT. The concept of Cayley-symmetric Γ- semigroups is introduced, and many equivalent conditions of a Cayley-symmetric Γ- semigroups are given. It is proved that a strong semilattice of self-decomposable Γ- semigroups S_α is Cayley-symmetric if and only if each S_α is Cayley-symmetric.

Key words: Generalized Cayley graphs, Cayley-Symmetric Γ- semigroup, strong semilattice of Γ- semigroups, self-decomposable.

I. INTRODUCTION

Based on the research papers on Cayley graphs of semigroups, Yongwen Zhu first introduced the concept of generalized Cayley graphs of semigroups in which some fundamental properties of generalized Cayley graphs of semigroups were studied. Based on the works on Cayley graphs of semigroups, we introduced the concept of Cayley-symmetric Γ- semigroups. Several equivalent conditions of a Cayley-symmetric Γ-semigroup are presented in this paper and established a necessary and sufficient condition for a semilattices of Γ- semigroups to be Cayley-symmetric.

II. PRELIMINARIES

Definition 2.1. Let T be an ideal extension of a semigroup S and $\rho \subseteq T^1 X T^1$, T^1 is a semigroup T with identity adjoined. Then Cayley graph $\mathrm{Cay}(S,\rho)$ of S relative to ρ is defined as the graph with vertex set S and edge set $E(\mathrm{Cay}(S,\rho))$ consisting of those ordered pairs (a,b), where $xay = b$ for some $(x,y) \in \rho$. Also we call these defined Cayley graphs as generalized Cayley graphs.

Notation 2.1. If S is a semigroup and $a \in S$, then $P(a) = S^1 a S^1, L(a) = S^1 a, R(a) = aS^1$ are the principal, left, right ideals generated by a resp. where S^1 is a semigroup with identity adjoined.

Definition 2.2. Let $S_L = S^1 \times \{1\}$, $S_R = \{1\} \times S^1$, $S_U = S^1 \times S^1$ be the left,right and the universal relations on S^1, then the generalized Cayley graphs $\mathrm{Cay}(S,S_L), \mathrm{Cay}(S,S_R), \mathrm{Cay}(S,S_U)$ are called the left universal, right universal and universal Cayley graphs of S resp.

Definition 2.3. A semigroup S is called Cayley-symmetric if $\mathrm{Cay}(S,S_L) = \mathrm{Cay}(S,S_R)$

Definition 2.4. Let T be an ideal extension of a semigroup S. If $\mathrm{Cay}(S,T_L) = \mathrm{Cay}(S,T_R)$, then we say that S is Cayley-symmetric in T.

Definition 2.4. Let $S = \{x, y, z, ...\}$ and $\Gamma = \{\alpha, \beta, \gamma, ...\}$ be two non-empty sets. Then S is called a Γ −semigroup if it satisfies (i) $x\gamma y \in S$ and (ii) $(x\beta y)\gamma z = x\beta(y\gamma z)$, for all $x, y, z \in S$ and $\beta, \gamma \in \Gamma$

Definition 2.5. A non-empty subset A of a Γ −semigroup S is called a Γ −subsemigroup of S if $A \Gamma A \subseteq A$.

Definition 2.6. A left (right) Γ-ideal of a Γ −semigroup S is a non-empty subset A of S such that $S \Gamma A \subseteq A$ $(A\Gamma S \subseteq A)$ and a two sided Γ −ideal or simply a Γ −ideal is that which is both a left and right Γ −ideal of S.

III. CAYLEY-SYMMETRIC Γ −SEMIGROUPS

Definition 3.1. Let T be an ideal extension of a Γ-semigroup S and $\rho \subseteq T^1 X T^1$, T^1 is a Γ -semigroup T with identity adjoined. Then Cayley graph $\mathrm{Cay}(S,\rho)$ of S relative to ρ is defined as the graph with vertex set S and edge set $E(\mathrm{Cay}(S,\rho))$ consisting of those ordered pairs (a,b), where $x\alpha a\beta y = b$ for some $(x,y) \in \rho$ and $\alpha, \beta \in \Gamma$

Notation 3.1. If T is an ideal extension of a Γ -semigroup S and $\subseteq S$, then $P_T(A), L_T(A)$ and $R_T(A)$ are the ideal, left ideal and right ideal generated by A.

Here $P_T(a) = T^1\Gamma a\Gamma T^1$, $L_T(a) = T^1\Gamma a$, $R_T(a) = a\Gamma T^1$ where $a \in A$ and

$$P(a) = S^1\Gamma a\Gamma S^1,\ L(a) = S^1\Gamma a,\ R(a) = a\Gamma S^1,$$
$a \in A$

Lemma 3.1.*If T is an ideal extension of a Γ −semigroup S, then the following are equivalent:*

(1) $L_T(a) = R_T(a)$ *for every* $a \in S$

(2) $L_T(a)$ *is a right ideal of T and* $R_T(a)$ *is a left ideal of T for every* $a \in S$

Definition 3.2. Let T be an ideal extension of a Γ −semigroup S. If $\mathrm{Cay}(S,T_L) = \mathrm{Cay}(S,T_R)$, then we say that S is Cayley-symmetric in T.

Theorem 3.2. *If T is an ideal extension of a Γ −semigroup S, then the following statements are equivalent:*

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(1) S is Cayley-symmetric in T

(2) $L_T(a) = R_T(a)$ *for every* $a \in S$

Proof. Suppose that S is Cayley-symmetric in T

From the definition of generalized Cayley graphs, we have $(a,b) \in E(\mathrm{Cay}(\mathrm{S},\mathrm{T_L}))$

$\Leftrightarrow b = x\alpha a$ for some $x \in \mathrm{T}^1, \alpha \in \Gamma$

$$\Leftrightarrow b \in \mathrm{T}^1\Gamma\mathrm{a} = \mathrm{L_T(a)}$$

Again $(a,b) \in E(\mathrm{Cay}(\mathrm{S},\mathrm{T_R}))$

$\Leftrightarrow b = a\beta y$ for some $y \in \mathrm{T}^1,\ \beta \in \Gamma$

$$\Leftrightarrow b \in \mathrm{a}\Gamma\,\mathrm{T}^1 = \mathrm{R_T(a)}$$

If $\mathrm{Cay}(\mathrm{S},\mathrm{T_L}) = \mathrm{Cay}(\mathrm{S},\mathrm{T_R})$, then for any $a,b \in S$, $(a,b) \in E(\mathrm{Cay}(\mathrm{S},\mathrm{T_L}))$ if and only if $(a,b) \in E(\mathrm{Cay}(\mathrm{S},\mathrm{T_R}))$. Thus for all $a,b \in S$ we have $b \in \mathrm{L_T(a)}$ if and only if $b \in \mathrm{R_T(a)}$.

As S is an ideal of T, $\mathrm{L_T(a)} = \mathrm{T}^1\Gamma\mathrm{a} \subseteq \mathrm{S}$ and $\mathrm{R_T(a)} = \mathrm{a}\Gamma\,\mathrm{T}^1 \subseteq \mathrm{S}$ for every $a \in S$.

Therefore $L_T(a) = R_T(a)$ for every a ∈ S.

Conversely, suppose that $L_T(a) = R_T(a)$ for every a ∈ S

Then for all $a,b \in S$, we have $b \in \mathrm{L_T(a)}$ if and only if $b \in \mathrm{R_T(a)}$.

Hence $(a,b) \in E(\mathrm{Cay}(\mathrm{S},\mathrm{T_L}))$ if and only if $(a,b) \in E(\mathrm{Cay}(\mathrm{S},\mathrm{T_R}))$.

i.e., $\mathrm{Cay}(\mathrm{S},\mathrm{T_L}) = \mathrm{Cay}(\mathrm{S},\mathrm{T_R})$ which implies S is Cayley-symmetric in T.

IV. Cayley-symmetry of strong semilattices of Γ- semigroups

First we define strong semilattice of Γ- semigroups as follows:

Let Y be a semilattice and a set of Γ- semigroups $S_\propto$ indexed by Y, and suppose that, for all $\alpha \geq \beta$ in Y there exists a mapping $\emptyset_{\alpha,\beta} : S_\propto \longrightarrow S_\beta$ such that : (1) for each $\alpha \in Y, \emptyset_{\alpha,\alpha} = 1_{S_\alpha}$

(2) $\emptyset_{\alpha,\beta}\,\emptyset_{\beta,\gamma} = \emptyset_{\alpha,\gamma}$ for all $\alpha,\beta,\gamma \in Y \ni \alpha \geq \beta \geq \gamma$. Let $S = \cup_{\alpha\in Y} S_\alpha$, the disjoint unions of $S_\alpha' s$. Define multiplication on S by $\mathrm{a}\,\Gamma\,\mathrm{b} = (\mathrm{a})\emptyset_{\alpha,\alpha\beta}\Gamma(\mathrm{b})\emptyset_{\beta,\alpha\beta}$. Then S is a Γ- semigroup, called the strong semilattice of Γ- semigroups $S_\propto$.

We write $S = S[Y;\ S_\alpha\ ;\ \emptyset_{\alpha,\beta}]$

Definition 4.1. Let S be a Γ- semigroup. If for every $a \in S, a \in S\Gamma\mathrm{a} \cap \mathrm{a}\Gamma\mathrm{S}$, then S is called self-decomposable.

Lemma 4.2. *Let S be a self-decomposable Γ- semigroup. Then S is Cayley-symmetric if and only if* $S\Gamma a = a\Gamma S$ *for all* $a \in S$.

Theorem 4.3. *Suppose that* $S = S[Y;\ S_\alpha\ ;\ \emptyset_{\alpha,\beta}]$, *where each* S_α *is self-decomposable. Then S is Cayley-symmetric if and only if for every* $\alpha \in Y$, S_α *is Cayley-symmetric.*

Proof. Since for each α, S_α is self-decomposable we have $S = S[Y;\ S_\alpha\ ;\ \emptyset_{\alpha,\beta}]$ is self-decomposable.

Necessary. Suppose that S is Cayley-symmetric. For $a,b \in S_\alpha$ with $\alpha \in Y$, by Lemma 4.2, there exist $\beta \in Y$ and $c \in S_\beta$ such that $a\alpha_1 b = c\alpha_2 a$; $\alpha_1,\alpha_2 \in \Gamma$. Since $a\alpha b \in S_\alpha$, we have $\beta \geq \alpha$. Now S_α is self-decomposable, there exists $\mathrm{u} \in S_\alpha$ such that $a = u\alpha_3 a;\ \alpha_3 \in \Gamma$. It gives that

$$a\alpha_1 b = c\alpha_2 a = c\alpha_2(u\alpha_3 a) = (c\alpha_2 u)\alpha_3 a,$$

where $c\alpha_2 u \in S_\alpha$.

Therefore $a\Gamma\, S_\alpha \subseteq S_\alpha\Gamma a$. Similarly we can show that $S_\alpha\Gamma a \subseteq a\Gamma\, S_\alpha$, which means that

$S_\alpha\Gamma a = a\Gamma\, S_\alpha$. By Lemma 5.2, S_α is Cayley-symmetric.

Sufficient. Suppose that S_α is Cayley-symmetric for all α. Let $a \in S_\alpha$ and $b \in S_\beta$ with $\alpha,\beta \in Y$

Take $\alpha\beta = \gamma \in Y$, then $\alpha,\beta \geq \gamma$. Since $a\sigma b \in S_\gamma;\ \sigma \in \Gamma$ and S_γ is self-decomposable, there exists $x,y \in S_\gamma$ such that $= x\tau(a\sigma b) = (a\sigma b)\varphi y;\ \tau,\varphi \in \Gamma$.

Therefore $a\sigma b = x\tau(a\sigma b)\varphi y = (x\tau a)\sigma(b\varphi y)$, where $x\tau a, b\varphi y \in S_\gamma$. Since S_γ is Cayley-symmetric, from Lemma 5.2 there exists $z \in S_\gamma$ such that $(x\tau a)\sigma(b\varphi y) = z\omega(x\tau a) = (z\omega x)\tau a$

This implies $a\sigma b = (z\omega x)\tau a \in S_a$. We have proved that $a\Gamma\mathrm{S} \subseteq \mathrm{S}\Gamma\mathrm{a}$.Similarly we can prove the inverse conclusion. Hence $a\Gamma\mathrm{S} = \mathrm{S}\Gamma\mathrm{a}$. Again by Lemma 5.2, S is Cayley-symmetric.

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