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On Bi-Orthogonal Bi-matrices

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Abstract - Bi-orthogonal bimatrices are studied as a generalization of orthogonal bimatrices. Some properties of bi-orthogonal bimatrices are discussed. Also, some results of bi-orthogonal bimatrices are obtained.

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I. INTRODUCTION

Matrices provide a very powerful tool for dealing with linear models. Bimatrices are still a powerful and an advanced tool which can handle over one linear model at a time. Bimatrices are useful when time bound comparisons are needed in the analysis of a model. Bimatrices are of several types. For $A \in C_{n\times n}, A^T, A^{-1}, A^{\dagger}$ and det (A) denote transpose, inverse, Moore-Penrose inverse and determinant of A respectively. If $AA^T = A^T A = I$ then A is said to be an orthogonal matrix, where I is an identity matrix.

Definition 1.1 [4]

A matrix A_B is said to be bimatrix then it is defined as the union of $A1$ and $A2$. That is, AB □□$A1$ □□$A2$. Here $A1$ and $A2$ be the component matrices. '□' is just the notational convenience (symbol) only.

Definition 1.2 [4]

Let $A_B^{m\times m} = A_1 \cup A_2$ be a mxm square bimatrix. We define $I_B^{m\times m} = I^{m\times m} \cup I^{m\times m} = I_1^{m\times m} \cup I_2^{m\times m}$ be the identity bimatrix.

Definition 1.3 [5]

Let a bimatrix A_B is said to be symmetric if its component matrices A_1 and A_2 are satisfied $A_1 = A_1^T$ and $A_2 = A_2^T$.

Definition 1.4 [5]

Let a bimatrix A_B is said to be skew-symmetric if its component matrices A_1 and A_2 are satisfied $A_1 = -A_1^T$ and $A_2 = -A_2^T$.

Definition 1.5 [3]

A bimatrix A_B is said to be orthogonal bimatrix, if $A_B A_B^T = A_B^T A_B = I_B$. That is, $A_B^T = A_B^{-1}$ (or) $\left(A_1^T \cup A_2^T\right) = \left(A_1^{-1} \cup A_2^{-1}\right)$.

In this paper, the concept of bi-orthogonal bimatrices is introduced. Some of the properties of orthogonal bimatrices are extended to bi-orthogonal bimatrices. Some results of bi-orthogonal bimatrices are obtained.

II. ON BI-ORTHOGONAL BI-MATRICES

The characterizations of bi-orthogonal bimatrices are studied in this section.

Definition 2.1

A bimatrix $A_B \in C_{n\times n}$ is bi-orthogonal bimatrix, if $A_B A_B^T A_B^T A_B = A_B^T A_B A_B A_B^T = I_B$

Example 2.2

Let $A_B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \cup \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ be the bi-orthogonal bimatrix.

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ie, $\left(A_1A_1^TA_1^TA_1 \cup A_2A_2^TA_2^TA_2\right) = \left(A_1^TA_1A_1A_1^T \cup A_2^TA_2A_2A_2^T\right) = \left(I_1 \cup I_2\right)$

Theorem 2.3

Product of two Bi orthogonal – Bi matrices of the same order is a Bi orthogonal – Bi matrix.

Proof

Let A_B and B_B be two Bi orthogonal – Bi matrices.

So that $\left(A_1A_1^TA_1^TA_1 \cup A_2A_2^TA_2^TA_2\right) = \left(A_1^TA_1A_1A_1^T \cup A_2^TA_2A_2A_2^T\right) = \left(I_1 \cup I_2\right)$

$\left(B_1B_1^TB_1^TB_1 \cup B_2B_2^TB_2^TB_2\right) = \left(B_1^TB_1B_1B_1^T \cup B_2^TB_2B_2B_2^T\right) = \left(I_1 \cup I_2\right)$

L.H.S

$$A_BA_B^TA_B^TA_B = I_B$$

$$B_BB_B^TB_B^TB_B = I_B$$

$$\left(A_BB_B\right)\left(A_BB_B\right)^T\left(A_BB_B\right)^T\left(A_BB_B\right) = I_B$$

$$= \left[\left(A_1 \cup A_2\right)\left(B_1 \cup B_2\right)\right]\left[\left(A_1 \cup A_2\right)\left(B_1 \cup B_2\right)\right]^T\left[\left(A_1 \cup A_2\right)\left(B_1 \cup B_2\right)\right]^T\left[\left(A_1 \cup A_2\right)\left(B_1 \cup B_2\right)\right]$$

$$= \left[A_1B_1 \cup A_2B_2\right]\left[A_1B_1 \cup A_2B_2\right]^T\left[A_1B_1 \cup A_2B_2\right]^T\left[A_1B_1 \cup A_2B_2\right]$$

$$= \left[A_1B_1 \cup A_2B_2\right]\left[\left(A_1B_1\right)^T \cup \left(A_2B_2\right)^T\right]\left[\left(A_1B_1\right)^T \cup \left(A_2B_2\right)^T\right]\left[A_1B_1 \cup A_2B_2\right]$$

$$= \left[A_1B_1 \cup A_2B_2\right]\left[B_1^TA_1^T \cup B_2^TA_2^T\right]\left[B_1^TA_1^T \cup B_2^TA_2^T\right]\left[A_1B_1 \cup A_2B_2\right]$$

$$= \left[A_1\left(B_1B_1^T\right)A_1^TB_1^T\left(A_1^TA_1\right)B_1\right] \cup \left[A_2\left(B_2B_2^T\right)A_2^TB_2^T\left(A_2^TA_2\right)B_2\right]$$

$$= \left[\left(A_1A_1^T\right)\left(B_1^TB_1\right)\right] \cup \left[\left(A_2A_2^T\right)\left(B_2^TB_2\right)\right]$$

$$= I_1 \cup I_2 \qquad \rightarrow (1)$$

R.H.S

$$A_B^TA_BA_BA_B^T = I_B$$

$$B_B^TB_BB_BB_B^T = I_B$$

$$\left(A_BB_B\right)^T\left(A_BB_B\right)\left(A_BB_B\right)\left(A_BB_B\right)^T = I_B$$

$$= \left[\left(A_1 \cup A_2\right)\left(B_1 \cup B_2\right)\right]^T\left[\left(A_1 \cup A_2\right)\left(B_1 \cup B_2\right)\right]\left[\left(A_1 \cup A_2\right)\left(B_1 \cup B_2\right)\right]\left[\left(A_1 \cup A_2\right)\left(B_1 \cup B_2\right)\right]^T$$

$$= \left[A_1B_1 \cup A_2B_2\right]^T\left[A_1B_1 \cup A_2B_2\right]\left[A_1B_1 \cup A_2B_2\right]\left[A_1B_1 \cup A_2B_2\right]^T$$

$$= \left[\left(A_1B_1\right)^T \cup \left(A_2B_2\right)^T\right]\left[A_1B_1 \cup A_2B_2\right]\left[A_1B_1 \cup A_2B_2\right]\left[\left(A_1B_1\right)^T \cup \left(A_2B_2\right)^T\right]$$

$$= \left[B_1^TA_1^T \cup B_2^TA_2^T\right]\left[A_1B_1 \cup A_2B_2\right]\left[A_1B_1 \cup A_2B_2\right]\left[B_1^TA_1^T \cup B_2^TA_2^T\right]$$

$$= \left[B_1^T\left(A_1^TA_1\right)B_1A_1\left(B_1B_1^T\right)A_1^T\right] \cup \left[B_2^T\left(A_2^TA_2\right)B_2A_2\left(B_2B_2^T\right)A_2^T\right]$$

$$= \left[\left(B_1^TB_1\right)\left(A_1A_1^T\right)\right] \cup \left[\left(B_2^TB_2\right)\left(A_2A_2^T\right)\right]$$

$$= I_1 \cup I_2 \qquad \rightarrow (2)$$

L.H.S = R.H.S

(1) = (2)

Since, $A_BA_B^TA_B^TA_B = A_B^TA_BA_BA_B^T = I_B$.

Theorem 2.4

Inverse of a Bi orthogonal – Bi matrix is a Bi orthogonal – Bi matrix

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Proof

For a Bi orthogonal – Bi matrix

$$A_B A_B^T A_B^T A_B = A_B^T A_B A_B A_B^T = I_B.$$

So that

$$A_B^{-1}\left(A_B^{-1}\right)^T\left(A_B^{-1}\right)^T A_B^{-1} = \left(A_B^{-1}\right)^T A_B^{-1} A_B^{-1}\left(A_B^{-1}\right)^T = I_B$$

L.H.S

$$A_B^{-1}\left(A_B^{-1}\right)^T\left(A_B^{-1}\right)^T A_B^{-1} = I_B$$

$$= \left(A_1 \cup A_2\right)^{-1}\left[\left(A_1 \cup A_2\right)^{-1}\right]^T\left[\left(A_1 \cup A_2\right)^{-1}\right]^T\left(A_1 \cup A_2\right)^{-1}$$

$$= \left[A_1^{-1} \cup A_2^{-1}\right]\left[A_1^{-1} \cup A_2^{-1}\right]^T\left[A_1^{-1} \cup A_2^{-1}\right]^T\left[A_1^{-1} \cup A_2^{-1}\right]$$

$$= \left[A_1^{-1} \cup A_2^{-1}\right]\left[\left(A_1^{-1}\right)^T \cup \left(A_2^{-1}\right)^T\right]\left[\left(A_1^{-1}\right)^T \cup \left(A_2^{-1}\right)^T\right]\left[A_1^{-1} \cup A_2^{-1}\right]$$

$$= \left[\left(A_1^{-1}\left(A_1^{-1}\right)^T\right)\left(\left(A_1^{-1}\right)^T A_1^{-1}\right)\right] \cup \left[\left(A_2^{-1}\left(A_2^{-1}\right)^T\right)\left(\left(A_2^{-1}\right)^T A_2^{-1}\right)\right]$$

$$= I_1 \cup I_2$$

$$= I_B \quad \rightarrow (1)$$

R.H.S

$$\left(A_B^{-1}\right)^T A_B^{-1} A_B^{-1}\left(A_B^{-1}\right)^T = I_B$$

$$= \left[\left(A_1 \cup A_2\right)^{-1}\right]^T\left(A_1 \cup A_2\right)^{-1}\left(A_1 \cup A_2\right)^{-1}\left[\left(A_1 \cup A_2\right)^{-1}\right]^T$$

$$= \left[A_1^{-1} \cup A_2^{-1}\right]^T\left[A_1^{-1} \cup A_2^{-1}\right]\left[A_1^{-1} \cup A_2^{-1}\right]\left[A_1^{-1} \cup A_2^{-1}\right]^T$$

$$= \left[\left(A_1^{-1}\right)^T \cup \left(A_2^{-1}\right)^T\right]\left[A_1^{-1} \cup A_2^{-1}\right]\left[A_1^{-1} \cup A_2^{-1}\right]\left[\left(A_1^{-1}\right)^T \cup \left(A_2^{-1}\right)^T\right]$$

$$= \left[\left(\left(A_1^{-1}\right)^T A_1^{-1}\right)\left(A_1^{-1}\left(A_1^{-1}\right)^T\right)\right] \cup \left[\left(\left(A_2^{-1}\right)^T A_2^{-1}\right)\left(A_2^{-1}\left(A_2^{-1}\right)^T\right)\right]$$

$$= I_1 \cup I_2$$

$$= I_B \quad \rightarrow (2)$$

L.H.S = R.H.S

(1) = (2)

$$A_B^{-1}\left(A_B^{-1}\right)^T\left(A_B^{-1}\right)^T A_B^{-1} = \left(A_B^{-1}\right)^T A_B^{-1} A_B^{-1}\left(A_B^{-1}\right)^T = I_B$$

Theorem 2.5

Transpose of a Bi orthogonal – Bi matrix is a Bi orthogonal – Bi matrix.

Proof

For a Bi orthogonal – Bi matrix

$$A_B A_B^T A_B^T A_B = A_B^T A_B A_B A_B^T = I_B$$

Now,

L.H.S

$$A_B^T\left(A_B^T\right)^T\left(A_B^T\right)^T A_B^T = I_B$$

$$= \left[A_1 \cup A_2\right]^T\left[A_1 \cup A_2\right]\left[A_1 \cup A_2\right]\left[A_1 \cup A_2\right]^T$$

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$$=[A_1 \cup A_2]^{\mathrm{T}}[A_1 \cup A_2][A_1 \cup A_2][A_1 \cup A_2]^{\mathrm{T}}$$
$$=\left[\left(A_1^{\mathrm{T}} A_1\right)\left(A_1 A_1^{\mathrm{T}}\right)\right] \cup\left[\left(A_2^{\mathrm{T}} A_2\right)\left(A_2 A_2^{\mathrm{T}}\right)\right]$$
$$=I_1 \cup I_2$$
$$=I_B \quad \rightarrow (1)$$

R.H.S

$$\left(A_B^{\mathrm{T}}\right)^{\mathrm{T}} A_B^{\mathrm{T}} A_B^{\mathrm{T}}\left(A_B^{\mathrm{T}}\right)^{\mathrm{T}}=I_B$$
$$=\left[\left(A_1 \cup A_2\right)\right][A_1 \cup A_2]^{\mathrm{T}}[A_1 \cup A_2]^{\mathrm{T}}\left[\left(A_1 \cup A_2\right)\right]$$
$$=[A_1 \cup A_2]\left[A_1^{\mathrm{T}} \cup A_2^{\mathrm{T}}\right]\left[A_1^{\mathrm{T}} \cup A_2^{\mathrm{T}}\right][A_1 \cup A_2]$$
$$=\left[\left(A_1 A_1^{\mathrm{T}}\right)\left(A_1^{\mathrm{T}} A_1\right)\right] \cup\left[\left(A_2 A_2^{\mathrm{T}}\right)\left(A_2^{\mathrm{T}} A_2\right)\right]$$
$$=I_1 \cup I_2$$
$$=I_B \quad \rightarrow (2)$$

L.H.S = R.H.S

(1) = (2)

$$A_B\left(A_B^{\mathrm{T}}\right)^{\mathrm{T}}\left(A_B^{\mathrm{T}}\right)^{\mathrm{T}} A_B=\left(A_B^{\mathrm{T}}\right)^{\mathrm{T}} A_B A_B\left(A_B^{\mathrm{T}}\right)^{\mathrm{T}}=I_B$$

Theorem 2.6

Transpose of a Bi orthogonal – Bi matrix.

Proof

Let $A_B = A_1 \cup A_2$ be a Bi orthogonal – Bi matrix.

That is

$$\left(A_1 A_1^T A_1^T A_1 \cup A_2 A_2^T A_2^T A_2\right)=\left(A_1^T A_1 A_1 A_1^T \cup A_2^T A_2 A_2 A_2^T\right)=\left(I_1 \cup I_2\right)$$

Now, consider

L.H.S

$$A_B A_B^{\mathrm{T}} A_B^{\mathrm{T}} A_B=A_B^{\mathrm{T}} A_B A_B A_B^{\mathrm{T}}=I_B$$
$$=[A_1 \cup A_2][A_1 \cup A_2]^{\mathrm{T}}[A_1 \cup A_2]^{\mathrm{T}}[A_1 \cup A_2]$$
$$=[A_1 \cup A_2]\left[A_1^{\mathrm{T}} \cup A_2^{\mathrm{T}}\right]\left[A_1^{\mathrm{T}} \cup A_2^{\mathrm{T}}\right][A_1 \cup A_2]$$
$$A_B A_B^{\mathrm{T}} A_B^{\mathrm{T}} A_B=\left[A_1 A_1^{\mathrm{T}} A_1^{\mathrm{T}} A_1\right] \cup\left[A_2 A_2^{\mathrm{T}} A_2^{\mathrm{T}} A_2\right]$$

Taking transpose on both sides

$$\left[A_B A_B^{\mathrm{T}} A_B^{\mathrm{T}} A_B\right]^{\mathrm{T}}=\left[A_1 A_1^{\mathrm{T}} A_1^{\mathrm{T}} A_1\right]^{\mathrm{T}} \cup\left[A_2 A_2^{\mathrm{T}} A_2^{\mathrm{T}} A_2\right]^{\mathrm{T}}$$
$$A_B^{\mathrm{T}}\left(A_B^{\mathrm{T}}\right)\left(A_B^{\mathrm{T}}\right) A_B=\left[A_1^{\mathrm{T}}\left(A_1^{\mathrm{T}}\right)^{\mathrm{T}}\left(A_1^{\mathrm{T}}\right)^{\mathrm{T}} A_1^{\mathrm{T}}\right] \cup\left[A_2^{\mathrm{T}}\left(A_2^{\mathrm{T}}\right)^{\mathrm{T}}\left(A_2^{\mathrm{T}}\right)^{\mathrm{T}} A_2^{\mathrm{T}}\right]$$
$$A_B A_B^{\mathrm{T}} A_B^{\mathrm{T}} A_B=\left[A_1 A_1^{\mathrm{T}} A_1^{\mathrm{T}} A_1\right] \cup\left[A_2 A_2^{\mathrm{T}} A_2^{\mathrm{T}} A_2\right]$$
$$A_B A_B^{\mathrm{T}} A_B^{\mathrm{T}} A_B=I_1 \cup I_2$$
$$A_B A_B^{\mathrm{T}} A_B^{\mathrm{T}} A_B=I_B \quad \rightarrow (1)$$

R.H.S

$$A_B^{\mathrm{T}} A_B A_B A_B^{\mathrm{T}}=I_B$$
$$A_B^{\mathrm{T}} A_B A_B A_B^{\mathrm{T}}=[A_1 \cup A_2]^{\mathrm{T}}[A_1 \cup A_2][A_1 \cup A_2][A_1 \cup A_2]^{\mathrm{T}}$$
$$A_B^{\mathrm{T}} A_B A_B A_B^{\mathrm{T}}=\left[A_1^{\mathrm{T}} \cup A_2^{\mathrm{T}}\right][A_1 \cup A_2][A_1 \cup A_2]\left[A_1^{\mathrm{T}} \cup A_2^{\mathrm{T}}\right]$$

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$$A_B^T A_B A_B A_B^T = \left[A_1^T A_1 A_1 A_1^T\right] \cup \left[A_2^T A_2 A_2 A_2^T\right]$$

Taking transpose on both sides

$$\left[A_B^T A_B A_B A_B^T\right]^T = \left[A_1^T A_1 A_1 A_1^T\right]^T \cup \left[A_2^T A_2 A_2 A_2^T\right]^T$$

$$\left[\left(A_B^T\right)^T A_B A_B^T \left(A_B^T\right)^T\right] = \left[\left(A_1^T\right)^T A_1^T A_1^T \left(A_1^T\right)^T\right] \cup \left[\left(A_2^T\right)^T A_2^T A_2^T \left(A_2^T\right)^T\right]$$

$$\left[A_B^T A_B A_B A_B^T\right] = \left[A_1^T A_1 A_1 A_1^T\right] \cup \left[A_2^T A_2 A_2 A_2^T\right]$$

$$A_B^T A_B A_B A_B^T = I_1 \cup I_2$$

$$A_B^T A_B A_B A_B^T = I_B \qquad \rightarrow (2)$$

L.H.S = R.H.S

(1) = (2)

$$A_B A_B^T A_B^T A_B = A_B^T A_B A_B A_B^T = I_B$$

Theorem 2.7

Any integral power of a Bi – Orthogonal Bi-Matrix is also a Bi orthogonal – Bi Matrix.

Proof

Let $A_B = A_1 \cup A_2$ be a Bi orthogonal – Bi matrix

$$A_B A_B^T A_B^T A_B = A_B^T A_B A_B A_B^T = I_B$$

Now,

L.H.S

Consider

$$A_B A_B^T A_B^T A_B = I_B$$

$$= [A_1 \cup A_2][A_1 \cup A_2]^T [A_1 \cup A_2]^T [A_1 \cup A_2]$$

$$= [A_1 \cup A_2]\left[A_1^T \cup A_2^T\right]\left[A_1^T \cup A_2^T\right][A_1 \cup A_2]$$

$$= \left[A_1 A_1^T A_1^T A_1\right] \cup \left[A_2 A_2^T A_2^T A_2\right]$$

$$= I_1 \cup I_2$$

$$= I_B$$

$$A_B A_B^T A_B^T A_B = I_B \qquad \rightarrow (1)$$

Again,

$$\left(A_B A_B^T A_B^T A_B\right)^2 = \left(A_B A_B^T A_B^T A_B\right)\left(A_B A_B^T A_B^T A_B\right)$$

$$\left(A_B A_B^T A_B^T A_B\right)^2 = \left(I_1 \cup I_2\right)\left(I_1 \cup I_2\right)$$

$$\left(A_B A_B^T A_B^T A_B\right)^2 = \left(I_1 \cup I_2\right)$$

$$\left(A_B A_B^T A_B^T A_B\right)^2 = I_B \qquad \rightarrow (2)$$

$\left(A_B A_B^T A_B^T A_B\right)^2$ is a Bi orthogonal – Bi matrix

R.H.S

$$A_B^T A_B A_B A_B^T = I_B$$

$$= [A_1 \cup A_2]^T [A_1 \cup A_2][A_1 \cup A_2][A_1 \cup A_2]^T$$

$$= \left[A_1^T \cup A_2^T\right][A_1 \cup A_2][A_1 \cup A_2]\left[A_1^T \cup A_2^T\right]$$

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$$=\left[A_1^{T}A_1A_1A_1^{T}\right]\cup\left[A_2^{T}A_2A_2A_2^{T}\right]$$
$$=I_1\cup I_2$$
$$=I_B$$
$$A_B^{T}A_BA_BA_B^{T}=I_B \qquad \rightarrow (1)$$

Again,

$$\left(A_B^{T}A_BA_BA_B^{T}\right)^2=\left(A_B^{T}A_BA_BA_B^{T}\right)\left(A_B^{T}A_BA_BA_B^{T}\right)$$
$$\left(A_B^{T}A_BA_BA_B^{T}\right)^2=\left(I_1\cup I_2\right)\left(I_1\cup I_2\right)$$
$$\left(A_B^{T}A_BA_BA_B^{T}\right)^2=\left(I_1\cup I_2\right)$$
$$\left(A_B^{T}A_BA_BA_B^{T}\right)^2=I_B \qquad \rightarrow (2)$$

$\left(A_B^{T}A_BA_BA_B^{T}\right)^2$ is a Bi orthogonal – Bi matrix

L.H.S = R.H.S

(1), (2)=(1), (2)

$$A_BA_B^{T}A_B^{T}A_B=A_B^{T}A_BA_BA_B^{T}=I_B$$
$$\left(A_BA_B^{T}A_B^{T}A_B\right)^2=\left(A_B^{T}A_BA_BA_B^{T}\right)^2=I_B$$

Hence, A_B^2 is a Bi orthogonal – Bi matrix

Assume that A_B^k is a Bi orthogonal – Bi matrix

That is,

$$\left(A_BA_B^{T}A_B^{T}A_B\right)^k=\left(A_B^{T}A_BA_BA_B^{T}\right)^k=I_B$$

To prove that A_B^{k+1} is a Bi orthogonal – Bi matrix.

L.H.S

$$\left(A_BA_B^{T}A_B^{T}A_B\right)^{k+1}=\left(A_BA_B^{T}A_B^{T}A_B\right)\left(A_BA_B^{T}A_B^{T}A_B\right)^k$$
$$=I_B.I_B$$
$$=I_B^2$$
$$\left(A_BA_B^{T}A_B^{T}A_B\right)^{k+1}=I_B \qquad \rightarrow (3)$$

R.H.S

$$\left(A_B^{T}A_BA_BA_B^{T}\right)^{k+1}=\left(A_B^{T}A_BA_BA_B^{T}\right)\left(A_B^{T}A_BA_BA_B^{T}\right)^k$$
$$=I_B.I_B$$
$$=I_B^2$$
$$\left(A_B^{T}A_BA_BA_B^{T}\right)^{k+1}=I_B \qquad \rightarrow (4)$$

L.H.S = R.H.S

(3) = (4)

$$\left(A_BA_B^{T}A_B^{T}A_B\right)^{k+1}=\left(A_B^{T}A_BA_BA_B^{T}\right)^{k+1}=I_B$$

Hence,

Any integral iower of a Bi orthogonal – Bi matrix is also a Bi orthogonal – Bi matrix.

Theorem 2.8

Let $A_B, B_B \in \square_{n\times n}$ be Bi orthogonal – Bi matrix,

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$$A_B B_B^{\mathrm{T}} B_B^{\mathrm{T}} A_B = B_B^{\mathrm{T}} A_B A_B B_B^{\mathrm{T}} \text{ and } B_B A_B^{\mathrm{T}} A_B^{\mathrm{T}} B_B = A_B^{\mathrm{T}} B_B B_B A_B^{\mathrm{T}}$$

(i) If $A_B B_B^{\mathrm{T}} B_B^{\mathrm{T}} A_B + B_B A_B^{\mathrm{T}} A_B^{\mathrm{T}} B_B = -I_B$, then $A_B + B_B$ is a Bi orthogonal – Bi matrix.

(ii) If $A_B B_B^{\mathrm{T}} B_B^{\mathrm{T}} A_B + B_B A_B^{\mathrm{T}} A_B^{\mathrm{T}} B_B = I_B$, then $A_B - B_B$ is a Bi orthogonal – Bi matrix.

Proof

Given that A_B and B_B is a Bi orthogonal – Bi matrices. Then we have

$$A_B A_B^{\mathrm{T}} A_B^{\mathrm{T}} A_B = A_B^{\mathrm{T}} A_B A_B A_B^{\mathrm{T}} = I_B \quad \text{and} \quad B_B B_B^{\mathrm{T}} B_B^{\mathrm{T}} B_B = B_B^{\mathrm{T}} B_B B_B B_B^{\mathrm{T}} = I_B$$

(i) Now

L.H.S

$$\begin{aligned}
(A_B A_B + B_B B_B)(A_B A_B + B_B B_B)^{\mathrm{T}} &= \left[(A_1 \cup A_2)(A_1 \cup A_2) + (B_1 \cup B_2)(B_1 \cup B_2)\right] \\
&\quad \left[(A_1 \cup A_2)(A_1 \cup A_2) + (B_1 \cup B_2)(B_1 \cup B_2)\right]^{\mathrm{T}} \\
&= \left[(A_1 A_1 + B_1 B_1) \cup (A_2 A_2 + B_2 B_2)\right] \quad \left[(A_1 A_1 + B_1 B_1) \cup (A_2 A_2 + B_2 B_2)\right]^{\mathrm{T}} \\
&= \left[(A_1 A_1 + B_1 B_1) \cup (A_2 A_2 + B_2 B_2)\right] \quad \left[(A_1 A_1 + B_1 B_1)^{\mathrm{T}} \cup (A_2 A_2 + B_2 B_2)^{\mathrm{T}}\right] \\
&= \left[(A_1 A_1 + B_1 B_1) \cup (A_2 A_2 + B_2 B_2)\right] \quad \left[(A_1^{\mathrm{T}} A_1^{\mathrm{T}} + B_1^{\mathrm{T}} B_1^{\mathrm{T}}) \cup (A_2^{\mathrm{T}} A_2^{\mathrm{T}} + B_2^{\mathrm{T}} B_2^{\mathrm{T}})\right] \\
&= \left[(A_1 A_1 + B_1 B_1)(A_1^{\mathrm{T}} A_1^{\mathrm{T}} + B_1^{\mathrm{T}} B_1^{\mathrm{T}})\right] \quad \left[(A_2 A_2 + B_2 B_2)(A_2^{\mathrm{T}} A_2^{\mathrm{T}} + B_2^{\mathrm{T}} B_2^{\mathrm{T}})\right] \\
&= \left[A_1 A_1 A_1^{\mathrm{T}} A_1^{\mathrm{T}} + A_1 A_1 B_1^{\mathrm{T}} B_1^{\mathrm{T}} + B_1 B_1 A_1^{\mathrm{T}} A_1^{\mathrm{T}} + B_1 B_1 B_1^{\mathrm{T}} B_1^{\mathrm{T}}\right] \\
&\quad \cup \left[A_2 A_2 A_2^{\mathrm{T}} A_2^{\mathrm{T}} + A_2 A_2 B_2^{\mathrm{T}} B_2^{\mathrm{T}} + B_2 B_2 A_2^{\mathrm{T}} A_2^{\mathrm{T}} + B_2 B_2 B_2^{\mathrm{T}} B_2^{\mathrm{T}}\right] \\
&= \left[I_1 + (A_1 A_1 B_1^{\mathrm{T}} B_1^{\mathrm{T}} + B_1 B_1 A_1^{\mathrm{T}} A_1^{\mathrm{T}}) + I_1\right] \cup \left[I_2 + (A_2 A_2 B_2^{\mathrm{T}} B_2^{\mathrm{T}} + B_2 B_2 A_2^{\mathrm{T}} A_2^{\mathrm{T}}) + I_2\right] \\
&= [I_1 - I_1 + I_1] \cup [I_2 - I_2 + I_2] \\
&= I_1 \cup I_2
\end{aligned}$$

$$(A_B A_B + B_B B_B)(A_B A_B + B_B B_B)^{\mathrm{T}} = I_B \quad \rightarrow (1)$$

R.H.S

$$\begin{aligned}
(A_B A_B + B_B B_B)^{\mathrm{T}} (A_B A_B + B_B B_B) &= \left[(A_1 \cup A_2)(A_1 \cup A_2) + (B_1 \cup B_2)(B_1 \cup B_2)\right]^{\mathrm{T}} \\
&\quad \left[(A_1 \cup A_2)(A_1 \cup A_2) + (B_1 \cup B_2)(B_1 \cup B_2)\right] \\
&= \left[(A_1 A_1 + B_1 B_1) \cup (A_2 A_2 + B_2 B_2)\right]^{\mathrm{T}} \cup \left[(A_1 A_1 + B_1 B_1) \cup (A_2 A_2 + B_2 B_2)\right] \\
&= \left[(A_1 A_1 + B_1 B_1)^{\mathrm{T}} \cup (A_2 A_2 + B_2 B_2)^{\mathrm{T}}\right] \left[(A_1 A_1 + B_1 B_1) \cup (A_2 A_2 + B_2 B_2)\right] \\
&= \left[(A_1^{\mathrm{T}} A_1^{\mathrm{T}} + B_1^{\mathrm{T}} B_1^{\mathrm{T}}) \cup (A_2^{\mathrm{T}} A_2^{\mathrm{T}} + B_2^{\mathrm{T}} B_2^{\mathrm{T}})\right] \left[(A_1 A_1 + B_1 B_1) \cup (A_2 A_2 + B_2 B_2)\right] \\
&= \left[(A_1^{\mathrm{T}} A_1^{\mathrm{T}} + B_1^{\mathrm{T}} B_1^{\mathrm{T}})(A_1 A_1 + B_1 B_1)\right] \cup \left[(A_2^{\mathrm{T}} A_2^{\mathrm{T}} + B_2^{\mathrm{T}} B_2^{\mathrm{T}})(A_2 A_2 + B_2 B_2)\right] \\
&= \left[A_1^{\mathrm{T}} A_1^{\mathrm{T}} A_1 A_1 + A_1^{\mathrm{T}} A_1^{\mathrm{T}} B_1 B_1 + B_1^{\mathrm{T}} B_1^{\mathrm{T}} A_1 A_1 + B_1^{\mathrm{T}} B_1^{\mathrm{T}} B_1 B_1\right] \\
&\quad \cup \left[A_2^{\mathrm{T}} A_2^{\mathrm{T}} A_2 A_2 + A_2^{\mathrm{T}} A_2^{\mathrm{T}} B_2 B_2 + B_2^{\mathrm{T}} B_2^{\mathrm{T}} A_2 A_2 + B_2^{\mathrm{T}} B_2^{\mathrm{T}} B_2 B_2\right] \\
&= \left[I_1 + (A_1^{\mathrm{T}} A_1^{\mathrm{T}} B_1 B_1 + B_1^{\mathrm{T}} B_1^{\mathrm{T}} A_1 A_1) + I_1\right] \cup \left[I_2 + (A_2^{\mathrm{T}} A_2^{\mathrm{T}} B_2 B_2 + B_2^{\mathrm{T}} B_2^{\mathrm{T}} A_2 A_2) + I_2\right] \\
&= [I_1 - I_1 + I_1] \cup [I_2 - I_2 + I_2] \\
&= I_1 \cup I_2
\end{aligned}$$

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$$\left(A_BA_B+B_BB_B\right)^{\mathrm{T}}\left(A_BA_B+B_BB_B\right)=I_B \qquad \to(2)$$

Hence, A_B+B_B is a Bi orthogonal – Bi matrix

(ii) Now,

L.H.S

$$\left(A_BA_B-B_BB_B\right)\left(A_BA_B-B_BB_B\right)^{\mathrm{T}}$$

$$=\left[\left(A_1\cup A_2\right)A_1\cup A_2-\left(B_1\cup B_2\right)\left(B_1\cup B_2\right)\right]\left[\left(A_1\cup A_2\right)A_1\cup A_2-\left(B_1\cup B_2\right)\left(B_1\cup B_2\right)\right]^{\mathrm{T}}$$

$$=\left[\left(A_1A_1-B_1B_1\right)\cup\left(A_2A_2-B_2B_2\right)\right]\left[\left(A_1A_1-B_1B_1\right)\cup\left(A_2A_2-B_2B_2\right)\right]^{\mathrm{T}}$$

$$=\left[\left(A_1A_1-B_1B_1\right)\cup\left(A_2A_2-B_2B_2\right)\right]\left[\left(A_1A_1-B_1B_1\right)^{\mathrm{T}}\cup\left(A_2A_2-B_2B_2\right)^{\mathrm{T}}\right]$$

$$=\left[\left(A_1A_1-B_1B_1\right)\left(A_1A_1-B_1B_1\right)^{\mathrm{T}}\right]\cup\left[\left(A_2A_2-B_2B_2\right)\left(A_2A_2-B_2B_2\right)^{\mathrm{T}}\right]$$

$$=\left[\left(A_1A_1-B_1B_1\right)\left(A_1^{\mathrm{T}}A_1^{\mathrm{T}}-B_1^{\mathrm{T}}B_1^{\mathrm{T}}\right)\right]\cup\left[\left(A_2A_2-B_2B_2\right)\left(A_2^{\mathrm{T}}A_2^{\mathrm{T}}-B_2^{\mathrm{T}}B_2^{\mathrm{T}}\right)\right]$$

$$=\left[A_1A_1A_1^{\mathrm{T}}A_1^{\mathrm{T}}-A_1A_1B_1^{\mathrm{T}}B_1^{\mathrm{T}}-B_1B_1A_1^{\mathrm{T}}A_1^{\mathrm{T}}+B_1B_1B_1^{\mathrm{T}}B_1^{\mathrm{T}}\right]$$
$$\cup\left[A_2A_2A_2^{\mathrm{T}}A_2^{\mathrm{T}}-A_2A_2B_2^{\mathrm{T}}B_2^{\mathrm{T}}-B_2B_2A_2^{\mathrm{T}}A_2^{\mathrm{T}}+B_2B_2B_2^{\mathrm{T}}B_2^{\mathrm{T}}\right]$$

$$=\left[I_1-\left(A_1A_1B_1^{\mathrm{T}}B_1^{\mathrm{T}}+B_1B_1A_1^{\mathrm{T}}A_1^{\mathrm{T}}\right)+I_1\right]\cup\left[I_2-\left(A_2A_2B_2^{\mathrm{T}}B_2^{\mathrm{T}}+B_2B_2A_2^{\mathrm{T}}A_2^{\mathrm{T}}\right)+I_2\right]$$

$$=\left[I_1-I_1+I_1\right]\cup\left[I_2-I_2+I_2\right]$$

$$=I_1\cup I_2$$

$$\left(A_BA_B-B_BB_B\right)\left(A_BA_B-B_BB_B\right)^{\mathrm{T}}=I_B \qquad \to(1)$$

R.H.S

$$\left(A_BA_B-B_BB_B\right)^{\mathrm{T}}\left(A_BA_B-B_BB_B\right)$$

$$=\left[\left(A_1\cup A_2\right)\left(A_1\cup A_2\right)-\left(B_1\cup B_2\right)\left(B_1\cup B_2\right)\right]^{\mathrm{T}}\left[\left(A_1\cup A_2\right)\left(A_1\cup A_2\right)-\left(B_1\cup B_2\right)\left(B_1\cup B_2\right)\right]$$

$$=\left[\left(A_1A_1-B_1B_1\right)\cup\left(A_2A_2-B_2B_2\right)\right]^{\mathrm{T}}\left[\left(A_1A_1-B_1B_1\right)\cup\left(A_2A_2-B_2B_2\right)\right]$$

$$=\left[\left(A_1A_1-B_1B_1\right)^{\mathrm{T}}\cup\left(A_2A_2-B_2B_2\right)^{\mathrm{T}}\right]\left[\left(A_1A_1-B_1B_1\right)\cup\left(A_2A_2-B_2B_2\right)\right]$$

$$=\left[\left(A_1^{\mathrm{T}}A_1^{\mathrm{T}}-B_1^{\mathrm{T}}B_1^{\mathrm{T}}\right)\cup\left(A_2^{\mathrm{T}}A_2^{\mathrm{T}}-B_2^{\mathrm{T}}B_2^{\mathrm{T}}\right)\right]\left[\left(A_1A_1-B_1B_1\right)\cup\left(A_2A_2-B_2B_2\right)\right]$$

$$=\left[\left(A_1^{\mathrm{T}}A_1^{\mathrm{T}}-B_1^{\mathrm{T}}B_1^{\mathrm{T}}\right)\left(A_1A_1-B_1B_1\right)\right]\cup\left[\left(A_2^{\mathrm{T}}A_2^{\mathrm{T}}-B_2^{\mathrm{T}}B_2^{\mathrm{T}}\right)\left(A_2A_2-B_2B_2\right)\right]$$

$$=\left[A_1^{\mathrm{T}}A_1^{\mathrm{T}}A_1A_1-A_1^{\mathrm{T}}A_1^{\mathrm{T}}B_1B_1-B_1^{\mathrm{T}}B_1^{\mathrm{T}}A_1A_1+B_1^{\mathrm{T}}B_1^{\mathrm{T}}B_1B_1\right]\cup\left[A_2^{\mathrm{T}}A_2^{\mathrm{T}}A_2A_2-A_2^{\mathrm{T}}A_2^{\mathrm{T}}B_2B_2-B_2^{\mathrm{T}}B_2^{\mathrm{T}}A_2A_2+B_2^{\mathrm{T}}B_2^{\mathrm{T}}B_2B_2\right]$$

$$=\left[I_1-\left(A_1^{\mathrm{T}}A_1^{\mathrm{T}}B_1B_1+B_1^{\mathrm{T}}B_1^{\mathrm{T}}A_1A_1\right)+I_1\right]\cup\left[I_2-\left(A_2^{\mathrm{T}}A_2^{\mathrm{T}}B_2B_2+B_2^{\mathrm{T}}B_2^{\mathrm{T}}A_2A_2\right)+I_2\right]=\left[I_1-I_1+I_1\right]\cup\left[I_2-I_2+I_2\right]=I_1\cup I_2$$

$$\left(A_BA_B-B_BB_B\right)^{\mathrm{T}}\left(A_BA_B-B_BB_B\right)=I_B \qquad \to(2)$$

L.H.S = R.H.S

(1) = (2)

$$\left(A_BA_B-B_BB_B\right)\left(A_BA_B-B_BB_B\right)^{\mathrm{T}}=\left(A_BA_B-B_BB_B\right)^{\mathrm{T}}\left(A_BA_B-B_BB_B\right)=I_B$$

Hence,

$\left(A_B-B_B\right)$ is a Bi orthogonal – Bi matrix

Theorem 2.9

If A_B is a Bi orthogonal – Bi matrix and λ is a real number, then λ A_B is a Bi orthogonal – Bi matrix.

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Proof

Given that A_B is a Bi orthogonal – Bi matrix.

That is,

$$A_B A_B^T A_B^T A_B = A_B^T A_B A_B A_B^T = I_B$$

Consider, $\lambda\left[A_B A_B^T A_B^T A_B\right] = I_B$

$$(\lambda A_B A_B)(\lambda A_B A_B)^T = \left[\lambda(A_1 \cup A_2)(A_1 \cup A_2)\right]\left[\lambda(A_1 \cup A_2)(A_1 \cup A_2)\right]^T$$

$$(\lambda A_B A_B)(\lambda A_B^T A_B^T) = [\lambda A_1 A_1 \cup \lambda A_2 A_2][\lambda A_1 A_1 \cup \lambda A_2 A_2]^T$$

$$\lambda^2 A_B A_B^T A_B^T A_B = [\lambda A_1 A_1 \cup \lambda A_2 A_2]\left[\lambda A_1^T A_1^T \cup \lambda A_2^T A_2^T\right]$$

$$= \left[(\lambda A_1 A_1)(\lambda A_1^T A_1^T) \cup (\lambda A_2 A_2)(\lambda A_2^T A_2^T)\right]$$

$$= \left[\lambda^2 A_1 A_1 A_1^T A_1^T \cup \lambda^2 A_2 A_2 A_2^T A_2^T\right]$$

$$= \left[\lambda^2 I_1 \cup \lambda^2 I_2\right]$$

$$= \lambda^2 (I_1 \cup I_2)$$

$$\lambda^2 A_B A_B^T A_B^T A_B = \lambda^2 I_B$$

$$(\lambda A_B A_B)(\lambda A_B A_B)^T = I_B \rightarrow (1)$$

R.H.S $\lambda\left[A_B^T A_B A_B A_B^T\right] = I_B$

$$(\lambda A_B A_B)^T (\lambda A_B A_B) = \left[\lambda(A_1 \cup A_2)(A_1 \cup A_2)\right]^T\left[\lambda(A_1 \cup A_2)(A_1 \cup A_2)\right]$$

$$(\lambda A_B^T A_B^T)(\lambda A_B A_B) = [\lambda A_1 A_1 \cup \lambda A_2 A_2]^T [\lambda A_1 A_1 \cup \lambda A_2 A_2]$$

$$(\lambda^2 A_B^T A_B^T A_B A_B) = \left[\lambda A_1^T A_1^T \cup \lambda A_2^T A_2^T\right][\lambda A_1 A_1 \cup \lambda A_2 A_2]$$

$$= \left[(\lambda A_1^T A_1^T)(\lambda A_1 A_1)\right] \cup \left[(\lambda A_2^T A_2^T)(\lambda A_2 A_2)\right]$$

$$= \left[\lambda^2 A_1^T A_1^T A_1 A_1\right] \cup \left[\lambda^2 A_2^T A_2^T A_2 A_2\right]$$

$$= \left[\lambda^2 I_1 \cup \lambda^2 I_2\right]$$

$$= \lambda^2 (I_1 \cup I_2)$$

$$\lambda^2 A_B^T A_B^T A_B A_B = \lambda^2 I_B$$

$$(\lambda A_B A_B)^T (\lambda A_B A_B) = I_B \rightarrow (2)$$

L.H.S=R.H.S

(1) = (2)

Hence, λA_B is a Bi orthogonal – Bi matrix

Theorem 2.10

Let $A_B \in C_{n\times n}$. If A_B is a Bi orthogonal – Bi matrix, then iA_B is a Bi orthogonal – Bi matrix.

Proof

Given that A_B is Bi orthogonal – Bi matrix.

That is,

$$A_B A_B^T A_B^T A_B = A_B^T A_B A_B A_B^T = I_B$$

L.H.S $A_B A_B^T A_B^T A_B = I_B$

$$(iA_B A_B)(iA_B A_B)^T = \left[i(A_1 \cup A_2)(A_1 \cup A_2)\right]\left[i(A_1 \cup A_2)(A_1 \cup A_2)\right]^T$$

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$$=[iA_1A_1 \cup iA_2A_2][iA_1A_1 \cup iA_2A_2]^{T}$$
$$=[iA_1A_1 \cup iA_2A_2]\left[iA_1^{T}A_1^{T} \cup iA_2^{T}A_2^{T}\right]$$
$$=\left[(iA_1A_1)(iA_1^{T}A_1^{T})\right] \cup \left[(iA_2A_2)(iA_2^{T}A_2^{T})\right]$$
$$=\left[i^2A_1A_1A_1^{T}A_1^{T}\right] \cup \left[i^2A_2A_2A_2^{T}A_2^{T}\right]$$
$$=I_1 \cup I_2$$
$$(iA_BA_B)(iA_BA_B)^{T}=I_B \quad \rightarrow (1)$$

R.H.S $\quad A_B^{T}A_BA_BA_B^{T}=I_B$

$$(iA_BA_B)^{T}(iA_BA_B)=\left[i(A_1 \cup A_2)(A_1 \cup A_2)\right]^{T}\left[i(A_1 \cup A_2)(A_1 \cup A_2)\right]$$
$$=[iA_1A_1 \cup iA_2A_2]^{T}[iA_1A_1 \cup iA_2A_2]$$
$$=\left[iA_1^{T}A_1^{T} \cup iA_2^{T}A_2^{T}\right][iA_1A_1 \cup iA_2A_2]$$
$$=\left[(iA_1^{T}A_1^{T})(iA_1A_1)\right] \cup \left[(iA_2^{T}A_2^{T})(iA_2A_2)\right]$$
$$=\left[i^2A_1^{T}A_1^{T}A_1A_1\right] \cup \left[i^2A_2^{T}A_2^{T}A_2A_2\right]$$
$$=I_1 \cup I_2$$
$$(iA_BA_B)^{T}(iA_BA_B)=I_B \quad \rightarrow (2)$$

L.H.S = R.H.S

(1) = (2)

$$(iA_BA_B)(iA_BA_B)^{T}=(iA_BA_B)^{T}(A_BA_B)=I_B$$

Thus, iA_B is Bi orthogonal – Bi matrix.

Theorem 2.11

A square bimatrix A_B is Bi orthogonal – Bi matrix if and only if $A_BA_B^{T}A_B^{T}A_B=A_B^{T}A_BA_BA_B^{T}=I_B$

Proof

If A_B is a Bi-orthogonal – Bi matrix.

Then $\quad A_B^{T}=A_B^{-1}$ and

$$A_BA_B^{T}A_B^{T}A_B=A_BA_B^{-1}A_B^{-1}A_B$$
$$=(A_1 \cup A_2)(A_1 \cup A_2)^{-1}(A_1 \cup A_2)^{-1}(A_1 \cup A_2)$$
$$=(A_1 \cup A_2)(A_1^{-1} \cup A_2^{-1})(A_1^{-1} \cup A_2^{-1})(A_1 \cup A_2)$$
$$=(A_1A_1^{-1})(A_1^{-1}A_1) \cup (A_2A_2^{-1})(A_2^{-1}A_2)$$
$$=I_1 \cup I_2$$
$$A_BA_B^{T}A_B^{T}A_B=I_B$$

Conversely, if $\quad A_BA_B^{T}A_B^{T}A_B=I_B$, then also

$$A_B^{T}A_BA_BA_B^{T}=A_B^{T}\left(A_B^{T}\right)^{T}\left(A_B^{T}\right)^{T}A_B^{T}$$
$$=\left(A_B^{T}A_BA_BA_B^{T}\right)^{T}$$
$$=\left[(A_1 \cup A_2)^{T}(A_1 \cup A_2)(A_1 \cup A_2)(A_1 \cup A_2)^{T}\right]^{T}$$
$$=\left[(A_1^{T} \cup A_2^{T})(A_1 \cup A_2)(A_1 \cup A_2)(A_1^{T} \cup A_2^{T})\right]^{T}$$
$$=\left[A_1^{T}A_1A_1A_1^{T} \cup A_2^{T}A_2A_2A_2^{T}\right]^{T}$$

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$$= \left[A_1^{\mathrm{T}} A_1 A_1 A_1^{\mathrm{T}} \cup A_2^{\mathrm{T}} A_2 A_2 A_2^{\mathrm{T}} \right]$$

$$= I_1 \cup I_2$$

$$= I_B$$

Hence,

$$A_B^{\mathrm{T}} = A_B^{-1}.$$

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