International Journal for Research in Engineering Application & Management (IJREAM)
ISSN : 2454-9150 Vol-05, Issue-01, April 2019

Functional Separable Method (FSM) and its Applications to Solve Nonlinear Partial Differential Equations

[1]D. A. Shah, [2]A. K. Parikh

[1]Research Scholar, [2]Principal, Mehsana Urban Institute of Sciences, Ganpat University, Ganpat Vidyanagar, Gujarat, India. [1]disha_154@yahoo.co.in, [2]amit.parikh.maths@gmail.com

Abstract **The present study discusses Functional Separable Method which can be used in solving nonlinear equations of mathematical physics. In this paper, our attempt is to deal with nonlinear problem using two different approaches. In first example, the analytical exact solution is obtained for nonstationary heat equation with general nonlinear source term by using Functional Separable solution in the linear form. Then in second example, the functional separable solution is obtained in the form of quadratic polynomial for nonstationary heat equation with specific logarithmic source term. It is worthwhile to note that many nonlinear partial differential equations that are not reducible to linear equations have exact solutions, such solutions are known as Functional Separable Solutions.**

Keywords - Functional Separable; Logarithmic source ; Nonlinear source ; Nonstationary heat equation

I. INTRODUCTION

Separation of variable method is the most common approach to solve linear equations of mathematical physics. For equations in two variables x, y and a dependent variable u, this approach involves searching for exact solutions in the form of the product of functions depending on different arguments [8] :

$$u(x,\ t)=p(x)q(t) \tag{1}$$

The integration of a few classes of first-order nonlinear partial differential equations is based on searching for exact solutions in the form of the sum of functions depending on different arguments [13] :

$$u(x,\ t)=p(x)+q(t) \tag{2}$$

Generalized separable solutions of nonlinear partial differential equations are determined by Parikh, A. K. [9]. Some second and higher order nonlinear equations of mathematical physics also have exact solutions of the form (1) or (2). Such solutions are known as multiplicative separable and additive separable, respectively [7]. We have discussed the solution of nonlinear partial differential equation using generalized Functional Separable Method and obtained the exact solution [10].

Suppose a nonlinear equation for $u = u(x, y)$ is found from a linear mathematical physics equation for $z = z(x, y)$ by a nonlinear change of variable $u=F(z)$. Then, if the linear equation for z admits separable solutions, the transformed nonlinear equation for u will have exact solutions; the transformed nonlinear equation for u will have exact solutions of the form [11]

$$u(x,y)=F(z), \quad \text{where } z=\sum_{m=1}^{n} p_m(x)q_m(y) \tag{3}$$

Many nonlinear partial differential equations that are not reducible to linear equations have exact solutions of the form (3) such solutions are known as functional separable solutions. In general, the functions $p_m(x)$, $q_m(y)$ and $F(z)$ in (3) are not known in advance and are to be identified.

Substituting (3) in the original partial differential equation, we obtain the functional-differential equation which can be reduce to the standard bilinear functional equation of the form

$$P_1(X)Q_1(Y)+P_2(X)Q_2(Y)+......+P_k(X)Q_k(Y)=0 \tag{4}$$

where

$$\left.\begin{aligned} P_j(X)&\equiv P_j(x,p_1,p_1',p_1'',.....,p_n,p_n',p_n''),\\ Q_j(Y)&\equiv Q_j(y,q_1,q_1',q_1'',.....,q_n,q_n',q_n'')\end{aligned}\right\}(j=1,\ 2,\ 3,\ ......,k) \tag{5}$$

Consider functional separable solutions of the form (3) in the special case where the composite argument z is linear in one of the independent variables (e.g., in x). We substitute (3) into the equation under study and exclude x using the expression of z to find a functional differential equation with two arguments.

 DOI : 10.18231/2454-9150.2019.0338

The simplest functional separable solution of special form (x and y can be swapped):

$$u = F(z), \quad z = q_1(y)x + q_2(y) \qquad (z \text{ is linear in } x)$$

is called a generalized travelling-wave solution. After substituting above expressions into the original equation, one should eliminate x with the help of the expression for z. This will result in a functional-differential equation with two arguments, y and z.

II. Algorithm For Constructing Generalized Functional Separable Solution

The scheme for constructing Generalized Functional Separable solutions for evolution equations is displayed in following figure.

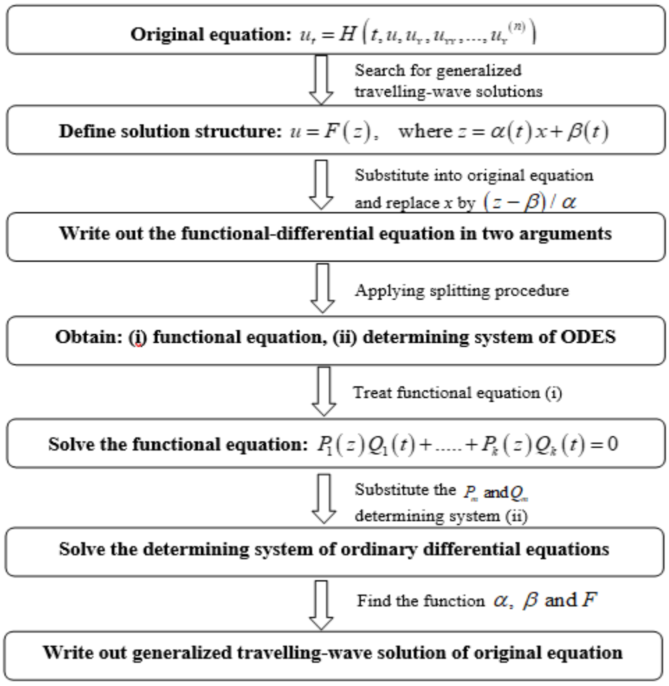

Figure - 1

We consider two examples of nonlinear equation that admits Functional Separable solutions of the special form where the argument z is linear.

III. Example – (A)

Consider the nonstationary heat equation with a nonlinear source [12]

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + F(u), \tag{6}$$

where $F(u)$ represents nonlinear source term.

IV. Functional Separable Solution

We seek functional separable solution in the special form

$$u = u(z), \quad z = \alpha(t)x + \beta(t) \tag{7}$$

Our purpose is to determine the unknown functions $u(z)$, $\alpha(t)$, $\beta(t)$ and $F(u)$.

By Substituting (7) into (6), we get

$$u'_z \cdot z'_t = \frac{\partial}{\partial x}\left[u'_z \cdot z'_x\right] + F(u)$$

On Simplifying, we get

$$\alpha_t{}' x + \beta_t{}' = \alpha^2 \frac{u''_{zz}}{u'_z} + \frac{F(u)}{u'_z} \tag{8}$$

Now we express $x = \dfrac{z - \beta(t)}{\alpha(t)}$ in terms of z and substitute in to (8) to obtain a functional-differential equation with two variables,

$$\alpha'_t\left[\frac{z-\beta(t)}{\alpha(t)}\right] + \beta'_t = \alpha^2 \frac{u''_{zz}}{u'_z} + \frac{F(u)}{u'_z}$$

This gives,

$$-\beta'_t + \frac{\beta}{\alpha}\alpha'_t - \frac{\alpha'_t z}{\alpha} + \alpha^2 \frac{u''_{zz}}{u'_z} + \frac{F(u)}{u'_z} = 0$$

This can be treated as the functional equation

$$\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 + \alpha_4\beta_4 = 0$$

where

$$\left.\begin{aligned} &\alpha_1 = -\beta'_t + \frac{\beta}{\alpha}\alpha'_t, \ \alpha_2 = \frac{-\alpha'_t}{\alpha}, \ \alpha_3 = \alpha^2, \ \alpha_4 = 1 \\ &\beta_1 = 1, \ \beta_2 = z, \ \beta_3 = \frac{u''_{zz}}{u'_z}, \ \beta_4 = \frac{F(u)}{u'_z} \end{aligned}\right\} \tag{9}$$

Putting these expressions (9) into following relations (10),

$$\left.\begin{aligned} \alpha_1 &= A_1\alpha_3 + A_2\alpha_4 \\ \alpha_2 &= A_3\alpha_3 + A_4\alpha_4 \\ \beta_3 &= -A_1\beta_1 - A_3\beta_2 \\ \beta_4 &= -A_2\beta_1 - A_4\beta_2 \end{aligned}\right\} [7] \tag{10}$$

We get the system of ordinary differential equations

$$\left.\begin{aligned} &-\beta'_t + \frac{\beta}{\alpha}\alpha'_t = A_1\alpha^2 + A_2, \quad \frac{-\alpha'_t}{\alpha} = A_3\alpha^2 + A_4 \\ &\frac{u''_{zz}}{u'_z} = -A_1 - A_3 z, \quad \frac{F(u)}{u'_z} = -A_2 - A_4 z \end{aligned}\right\} \tag{11}$$

where A_i ; $i = 1,2,3,4$ are arbitrary constants.

Now we consider two cases for A_4 :

Case 1. For $A_4 \neq 0$ the solution of system (11) is given by

Reducing first equation of (11) in to simplified form,

$$-\frac{d}{dt}\left(\frac{\beta}{\alpha}\right) = A_1\alpha + \frac{A_2}{\alpha}$$

Integrating,

$$\beta(t) = -\alpha(t)\left[A_1\int \alpha(t)\,dt + A_2\int \frac{dt}{\alpha(t)} + C_2\right]$$

Second equation of (11) can be rewritten as,

$$\frac{d\alpha}{dt} + A_4\alpha = -A_3\alpha^3$$

 DOI : 10.18231/2454-9150.2019.0338

By simplification, we get

$$\alpha^{-3}\frac{d\alpha}{dt}+\alpha^{-2}A_4=-A_3$$

For more simplification, taking $\alpha^{-2}=v$, we get

$$\frac{dv}{dt}-2vA_4=2A_3$$

This is a first order linear differential equation.

Now, integrating factor is e^{-2A_4t} .

Solution is given by,

$v\cdot e^{-2A_4t}=\int 2A_3\cdot e^{-2A_4t}dt+C_1$ which gives,

$$\alpha^{-2}=\frac{-A_3}{A_4}+C_1e^{2A_4t}$$

Therefore, the solution for $\alpha(t)$ can be written as

$$\alpha(t)=\pm\left(C_1e^{2A_4t}-\frac{A_3}{A_4}\right)^{-1/2}$$

By integrating third equation of (11),

$$\log u'_z=-A_1z-A_3\frac{z^2}{2}+C_3$$

On simplifying, we get

$$u'_z=e^{\left(\frac{-1}{2}A_3z^2-A_1z\right)+C_3}$$

Again integrating, we get

$$u(z)=C_3\int\exp\left(\frac{-1}{2}A_3z^2-A_1z\right)+C_4$$

Using fourth equation of (11),

$$F(u)=u'_z\left[-(A_2+A_4z)\right]$$

Substituting $u'_z=e^{\left(-\frac{1}{2}A_3z^2-A_1z\right)+c_3}$ in the above equation, we get

$$F(u)=-(A_2+A_4z)\cdot\left[C_3\int\exp\left(\frac{-1}{2}A_3z^2-A_1z\right)\right]$$

Thus for $A_4\neq 0$ the solution of system (11) is given by

$$\left.\begin{aligned}
&\beta(t)=-\alpha(t)\left[A_1\int\alpha(t)dt+A_2\int\frac{dt}{\alpha(t)}+C_2\right]\\
&\alpha(t)=\pm\left(C_1e^{2A_4t}-\frac{A_3}{A_4}\right)^{-1/2}\\
&u(z)=C_3\int\exp\left(\frac{-1}{2}A_3z^2-A_1z\right)+C_4\\
&F(u)=-(A_2+A_4z)\cdot\left[C_3\int\exp\left(\frac{-1}{2}A_3z^2-A_1z\right)\right]
\end{aligned}\right\}\qquad(12)$$

where C_i ; $i=1,2,3,4$ are arbitrary constants.

The dependence $F=F(u)$ is defined by the last two relations in parametric form (z is considered the parameter).

If $A_3\neq 0$, in (11) the source function is expressed in terms of the inverse of the error function.

In special case $A_3=C_4=0,\ A_1=-1$ and $C_3=1$ the source function can be represented in explicit form as

$$F(w)=-w(A_4\ln w+A_2)\qquad(13)$$

Solutions of equation (7) in this case were obtained by [1] with group theoretic methods.

Case 2. For $A_4=0$, the solution to the first two equation in (11) is given by,

Reducing second equation of (11) in to simplified form,

$$-\alpha^{-3}\ d\alpha=A_3\ dt$$

Integrating we get solution for $\alpha(t)$,

$$\alpha(t)=\pm\frac{1}{\sqrt{2A_3t+C_1}}\qquad(14)$$

Now using first equation of (12),

$$\beta(t)=-\alpha(t)\left[A_1\int\alpha(t)dt+A_2\int\frac{dt}{\alpha(t)}+C_2\right]$$

Therefore we get,

$$\beta(t)=\frac{1}{\sqrt{2A_3t+C_1}}\left[-A_1\int(2A_3t+C_1)^{-1/2}dt-A_2\int(2A_3t+C_1)^{1/2}dt+C_2\right]$$

On simplifying, we get

$$\beta(t)=\frac{-A_1}{A_3}-\frac{A_2}{3A_3}(2A_3t+C_1)+\frac{C_2}{\sqrt{2A_3t+C_1}}\qquad(15)$$

and the solutions to the other equations are determined by the last formulas in (11) where $A_4=0$.

V. Example – (B)

Consider the nonlinear heat equation with a logarithmic source [3]

$$\frac{\partial u}{\partial t}=a\frac{\partial^2u}{\partial x^2}+f(t)u\ln u+g(t)u\qquad(16)$$

VI. Functional Separable Solution

By the transform of variable $u=e^z$ to the quadratically nonlinear equation [6]

$$\frac{\partial z}{\partial t}=a\frac{\partial^2z}{\partial x^2}+a\left(\frac{\partial z}{\partial x}\right)^2+f(t)z+g(t)\qquad(17)$$

We seek its functional separable solution in the form [4, 5]

$$z=\alpha_1(x)\beta_1(t)+\alpha_2(x)\beta_2(t)+\beta_3(t)$$

where $\alpha_1(x)=x^2$ and $\alpha_2(x)=x$ (18)

Inserting (18) in to equation (17), we obtain

 DOI : 10.18231/2454-9150.2019.0338

$$x^2\beta'_1(t)+x\beta'_2(t)+\beta'_3(t)=2a\beta_1(t)+4ax^2\beta_1^2(t)+4ax\beta_1(t)\beta_2(t)+a\beta_2^2(t)+f(t)x^2\beta_1(t)+f(t)x\beta_2(t)+f(t)\beta_3(t)+g(t)$$

By simplification we obtain,

$$x^2\beta_1'(t)+x\beta'_2(t)+\beta'_3(t)=x^2\left[4a\beta_1^2(t)+f(t)\beta_1(t)\right]+x\left[4a\beta_1(t)\beta_2(t)+f(t)\beta_2(t)\right]+2a\beta_1(t)+a\beta_2^2(t)+f(t)\beta_3(t)+g(t)$$

On equating like powers of *x* we get,

$$\beta_1'(t)=4a\beta_1^2(t)+f(t)\beta_1(t) \tag{19}$$

$$\beta_2'(t)=4a\beta_1(t)\beta_2(t)+f(t)\beta_2(t) \tag{20}$$

$$\beta_3'(t)=2a\beta_1(t)+a\beta_2'(t)+f(t)\beta_3(t)+g(t) \tag{21}$$

Using equation (19),

$$\frac{\partial\beta_1}{\partial t}-f(t)\beta_1(t)=4a\beta_1^2(t)$$

We can write this equation as,

$$\beta_1^{-2}(t)\frac{\partial\beta_1}{\partial t}-\beta_1^{-1}(t)f(t)=4a$$

Putting $-\beta_1^{-1}(t)=v$ and $\beta_1^{-2}\frac{\partial\beta_1}{\partial t}=\frac{\partial v}{\partial t}$ we obtain,

$\frac{\partial v}{\partial t}+vf(t)=4a$ which is a first order linear differential equation [2].

Now, integrating factor is $e^{\int f(t)\,dt}=F(t)$ (say)

Solution is given by,

$$v=\frac{4a}{F(t)}\int F(t)\,dt+\frac{c_1}{F(t)}$$ which gives

$$-\frac{1}{\beta_1(t)}=\frac{4a}{F(t)}\int F(t)\,dt+\frac{c_1}{F(t)}$$

By simplifying,

$$\beta_1(t)=\frac{1}{-\frac{4a}{F(t)}\int F(t)\,dt+\frac{c_1}{F(t)}} \tag{22}$$

Using equation (19),

$$\frac{\beta'_2(t)}{\beta_2(t)}=4a\beta_1(t)+f(t)$$

On integrating, we obtain

$$\log\beta_2(t)=4a\int\beta_1(t)\,dt+\int f(t)\,dt+c_2$$

By simplifying, we obtain

$$\beta_2(t)=e^{4a\int\beta_1(t)\,dt+\int f(t)\,dt+c_2}$$

On Substituting (22) in to the above equation, we obtain

$$\beta_2(t)=e^{\int\left[\frac{4a}{-\frac{4a}{F(t)}\int F(t)\,dt+\frac{c_1}{F(t)}}+f(t)\right]dt+c_2} \tag{23}$$

Using equation (20),

$$\frac{\partial\beta_3(t)}{\partial t}-f(t)\beta_3(t)=2a\beta_1(t)+a\beta_2^2(t)+g(t)$$

Now, integrating factor is $e^{-\int f(t)\,dt}=\frac{1}{F(t)}$

Solution is given by,

$$\beta_3(t)\frac{1}{F(t)}=\int\left\{\left[\frac{2a}{-\frac{4a}{F(t)}\int F(t)\,dt+\frac{c_1}{F(t)}}\right]+a\left[e^{\int\left(\frac{4a}{-\frac{4a}{F(t)}\int F(t)dt+\frac{c_1}{F(t)}}+f(t)\right)dt+c_2}\right]^2+g(t)\right\}\frac{1}{F(t)}\,dt+c_3$$

By simplification we obtain,

$$\beta_3(t)=F(t)\int\left\{\left[\frac{2a}{-\frac{4a}{F(t)}\int F(t)\,dt+\frac{c_1}{F(t)}}\right]+a\left[e^{\int\left(\frac{4a}{-\frac{4a}{F(t)}\int F(t)dt+\frac{c_1}{F(t)}}+f(t)\right)dt+c_2}\right]^2+g(t)\right\}\frac{1}{F(t)}\,dt+c \tag{24}$$

VII. CONCLUSION

Here we have discussed Generalized Functional Separable Method for special type of functional equations that arises most frequently in mathematical physics. For the first problem, Functional Separable solution in the form of linear expression for nonstationary heat equation (6) has been obtained in (12) where different cases have been discussed

 DOI : 10.18231/2454-9150.2019.0338

for different parameters which results into different forms for the source function. For second problem related to nonlinear heat equation with logarithmic source (16), the Functional Separable solution is obtained in (22) to (24) in the form of transcendental function. The Functional Separable Method also used for constructing exact solutions for some classes of nonlinear heat and wave equations.

VIII. ACKNOWLEDGEMENT

I am very thankful to C. U. Shah University and Ganpat University for giving me an opportunity to do research work in this area.

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 DOI : 10.18231/2454-9150.2019.0338