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# Numerical Study For Solving Fourth Order Ordinary Differential Equations

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**Abstract: Fourth order ordinary differential equations have many applications in science and engineering. Several numerical methods have been developed by the researchers in order to find the solutions of various problems arising in science and engineering. In this paper, we are discussing some classical numerical methods for solving such equations and comparing their accuracy to identify the most relevant method. The numerical solutions are in good agreement with the exact solutions. Point-wise absolute errors are obtained by using MATLAB software. Numerical results show the accuracy of the present numerical methods.**

***Keywords* — Laplace transform method, Adomian decomposition method, Initial value problems, Fourth order ordinary differential equations.**

## I. INTRODUCTION

Several problems in science and engineering can be formula ted in terms of linear and nonlinear differential equations. Linear and nonlinear differential equations are equations involving a relation between an unknown function and one or more of its derivatives. Equations involving derivatives of only one independent variable are called ordinary differential equations and may be classified as either initial value problem (IVP) or boundary value problem (BVP). Many authors attempt to solve such type of problems to obtain high accurate numerical solutions. Different numerical schemes have been established for solving such type of problems such as Taylor series method, Euler method, Runge-Kutta method, Picard method, finite difference method. The knowledge of Laplace transforms has in recent years become an essential part of mathematical background required of engineers and scientists. This is because the transform methods provide an easy and effective means for the solution of many problems arising in engineering. The method of Laplace transforms has the advantage of directly giving the solution of differential equations with given boundary values without the necessity of first finding the general solution and then evaluating from it the arbitrary constants. Initial value problems for solving ordinary differential equations have been presented in [1]. Numerical solutions of quadratic Riccati equations have been presented in [2]. Many computational methods for solving differential equations have been presented in [4]. The inverse Laplace transform has been used for eternal solutions of the Boltzmann equation in [8]. Numerical methods for solving fourth-order fractional integro-differential equations have been presented in [9]. An optimal homotopy analysis method has been used for solving nonlinear fractional differential equations in [11].

## II. FOURTH-ORDER ODE

Consider the general fourth order initial value problems (IVP) of the form:

$$y^{(iv)}+a(x)y'''+b(x)y''+c(x)y'+d(x)y = f(x), \quad 0\le x\le L \qquad (1)$$

with the initial conditions

$$y(0)=\alpha,\ y'(0)=\beta,\ y''(0)=\gamma \text{ and } y'''(0)=\delta \qquad (2)$$

where $a(x),\ b(x),\ c(x)$ and $d(x)$ are continuous functions of $x$ and $\alpha,\ \beta,\ \gamma$ and $\delta$ are constants.

## III. LAPLACE TRANSFORM:

Let $f(t)$ be a function of $t$ defined for all positive values of $t$. Then, the Laplace transforms of $f(t)$, denoted by $L\{f(t)\}$ is defined as:

$$L\{f(t)\}=\int_0^\infty e^{-st}f(t)dt,$$

provided that the integral exists. $s$ is a parameter which may be a real or complex number. The inverse Laplace transform of $f(t)$ is:

$$f(t)=L^{-1}\left\{\int_0^\infty e^{-st}f(t)dt\right\}$$

## IV. LAPLACE TRANSFORM METHOD FOR SOLVING FOURTH-ORDER ODES

Consider the fourth-order ordinary differential equation (1).

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Taking Laplace transform both sides of (1), we get

$$L\{y^{(iv)}\}+L\{a(x)y'''\}+L\{b(x)y''\}+L\{c(x)y'\}$$

$$+L\{d(x)y\}=L\{f(x)\}$$

$$[s^4\bar{y}(s)-s^3y(0)-s^2y'(0)-sy''(0)-y'''(0)]$$

$$+a[s^3\bar{y}(s)-s^2y(0)-sy'(0)-y''(0)]$$

$$+b[s^2\bar{y}(s)-sy(0)-y'(0)]+c[s\bar{y}(s)-y(0)]$$

$$+d\bar{y}(s)=\bar{f}(s)$$

Applying initial conditions and after some modifications, we get

$$[s^4+as^3+bs^2+cs+d]\bar{y}(s)=[s^3\alpha+s^2\beta+s\gamma+\delta]$$

$$+[as^2\alpha+as\beta+a\gamma]+[bs\alpha+b\beta+c\alpha]+\bar{f}(s)$$

Taking inverse Laplace transform, we get the required solution.

## V. ADOMIAN DECOMPOSITION METHOD

Adomian decomposition method is a powerful numerical technique for solving linear and nonlinear differential equations. Consider the general fourth order ordinary differential equation

$$\varphi^{(iv)}(x)=g(x,\varphi(x),\varphi'(x),\varphi''(x),\varphi'''(x)), \qquad (3)$$

with initial conditions

$$\varphi(0)=\alpha_1,\varphi'(0)=\alpha_2,\varphi''(0)=\alpha_3,\varphi'''(0)=\alpha_4$$

Equation (3) can be written as

$$D(\varphi)=e, \qquad (4)$$

where D is the differential operator, $\varphi$ and e are functions of $x$. Equation (4) can be written in operator form as

$$A\varphi+B\varphi+C\varphi=e,$$

where A is linear term, which is invertible. B is the remaining linear terms and C represents the nonlinear terms. Solving for $A\varphi$ as

$$A\varphi=e-B\varphi-C\varphi$$

Since A is invertible. Above equation can be written as

$$A^{-1}A\varphi=A^{-1}e-A^{-1}B\varphi-A^{-1}C\varphi \qquad (5)$$

where $A^{-1}$ is a four-fold integral operator and is defined as

$$A^{-1}(.)=\int_0^x\int_0^x\int_0^x\int_0^x(.)dx\,dx\,dx\,dx.$$

Equation (5) becomes

$$\varphi=\alpha_1+\alpha_2t+\frac{\alpha_3t^2}{2}+\frac{\alpha_4t^3}{6}+A^{-1}e-A^{-1}B\varphi-A^{-1}C\varphi \qquad (6)$$

The function $\varphi$ is decomposed into a series defined as

$$\varphi(x)=\sum_{n=0}^{\infty}\varphi_n(x),$$

with $\varphi_0$ identified as the first five terms on the right hand side of (6). The nonlinear term which is decomposed into Adomian polynomials as

$$g(x,\varphi)=\sum_{n=0}^{\infty}A_n$$

is considered as zero as there is no nonlinear terms exist in the given equation.

$$\varphi_1=-A^{-1}B\varphi_0,$$

$$\varphi_2=-A^{-1}B\varphi_1,$$

$$\varphi_3=-A^{-1}B\varphi_2,$$

$$\varphi_4=-A^{-1}B\varphi_3,$$

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$$\varphi_{n+1}=-A^{-1}B\varphi_n,$$

This series converges when the nth partial sum

$$\varphi_n=\sum_{i=0}^{n-1}\varphi_i$$

will be the approximate solution.

## VI. NUMERICAL EXAMPLES

The Laplace transform method of solving differential equations yields particular solutions without the necessary of first finding the general solution and then evaluating the arbitrary constants. Laplace transform method is especially useful for solving linear differential equations with constant coefficients.

**Example 1:** Consider the fourth order initial value problem

$$y^{iv}=x, \qquad 0\le x\le 1 \qquad (7)$$

subject to the initial conditions

$$y(0)=0,\quad y'(0)=1,\quad y''(0)=0,\quad y'''(0)=0.$$

The exact solution of the problem is

$$y(x)=\frac{x^5}{120}+x.$$

Taking Laplace transform both sides of (7), we get

$$[s^4\bar{y}(s)-s^3y(0)-s^2y'(0)-sy''(0)-y'''(0)]=\frac{1}{s^2}$$

Apply initial conditions, we obtain

$$[s^4\bar{y}(s)-s^2]=\frac{1}{s^2}$$

$$\bar{y}(s)=\frac{1}{s^6}+\frac{1}{s^2},$$

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Taking Inverse Laplace transform, we get

$$y(x) = \frac{x^5}{120} + x.$$

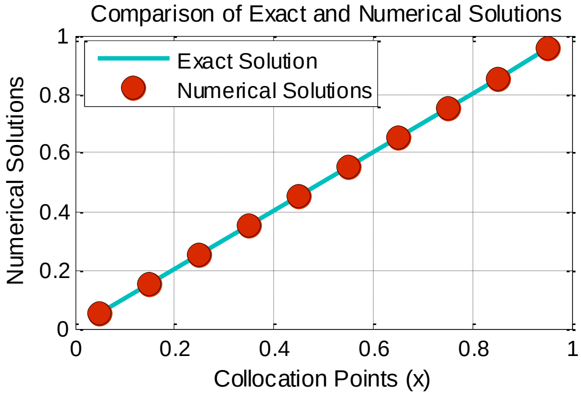


**Figure 1: Comparison of exact and numerical solutions**

Figure 1 shows the comparison of exact and numerical solutions obtained by Laplace transform method for Example 1.

**Example 2:** Consider the fourth order initial value problem

$$y^{(iv)} = e^{-2x} + sinx, \tag{8}$$

subject to the initial conditions

$$y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 0, \quad y'''(0) = 0.$$

The exact solution of the problem is

$$y(x) = \frac{e^{-2x}}{16} - \frac{1}{16} - \frac{7x}{8} - \frac{x^2}{8} + \frac{x^3}{4} + sinx.$$

Taking Laplace transform both side of (8), we get

$$s^4\bar{y}(s) - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) = \frac{1}{s+2} + \frac{1}{s^2+1},$$

Applying initial conditions, we get

$$\bar{y}(s) = \frac{1}{s^4(s+2)} + \frac{1}{s^4(s^2+1)}, \tag{9}$$

As we know

$$L^{-1}\left[\frac{f(s)}{s}\right] = \int_0^x f(x)dx.$$

Taking inverse Laplace transform in Equation (9) and using above formula, we get after simplifications

$$y(x) = \frac{e^{-2x}}{16} - \frac{1}{16} - \frac{7x}{8} - \frac{x^2}{8} + \frac{x^3}{4} + sinx.$$

Figure 2 shows the comparison of exact and numerical solution of Example 2.

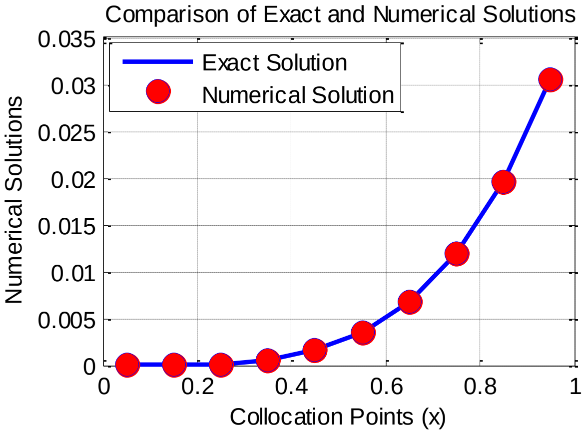


**Figure 2: Comparison of exact and numerical solutions**

**Example 3:** Consider the fourth order initial value problem

$$y^{iv} - k^4 y = 0, \quad 0 \le x \le 1 \tag{10}$$

subject to the conditions

$$y(0) = 1, \quad y'(0) = y''(0) = y'''(0) = 0.$$

The exact solution of the problem is

$$y(x) = \frac{1}{2}[\cos kt + \cosh kt]$$

Taking Laplace transform both sides of (10)

$$[s^4\bar{y}(s) - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0)] - k^4\{\bar{y}\} = 0$$

$$[s^4\bar{y}(s) - s^3] - k^4\{\bar{y}(s)\} = 0$$

$$\bar{y}(s) = \frac{s^3}{s^4 - k^4}.$$

Taking inverse Laplace transform, we get

$$y(x) = L^{-1}\left\{\frac{s^3}{s^4 - k^4}\right\},$$

$$y(x) = L^{-1}\left\{\frac{1}{s\left(1 - \frac{k^4}{s^4}\right)}\right\},$$

$$y(x) = L^{-1}\left\{\frac{1}{s}\left(1 + \frac{k^4}{s^4} + \frac{k^8}{s^8} + \frac{k^{12}}{s^{12}} + - - - - - -\right)\right\},$$

$$y(x) = L^{-1}\left\{\frac{1}{s} + \frac{k^4}{s^5} + \frac{k^8}{s^9} + \frac{k^{12}}{s^{13}} + - - - - - -\right\},$$

$$y(x) = 1 + \frac{(kx)^4}{4!} + \frac{(kx)^8}{8!} + \frac{(kx)^{12}}{12!} + - - - - - - -.$$


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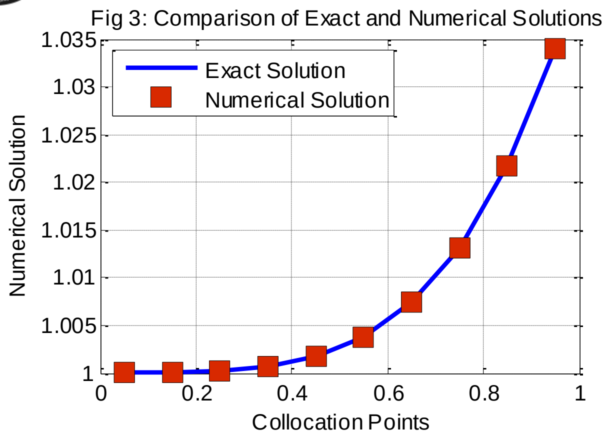


**Figure 3: Comparison of exact and numerical solutions**

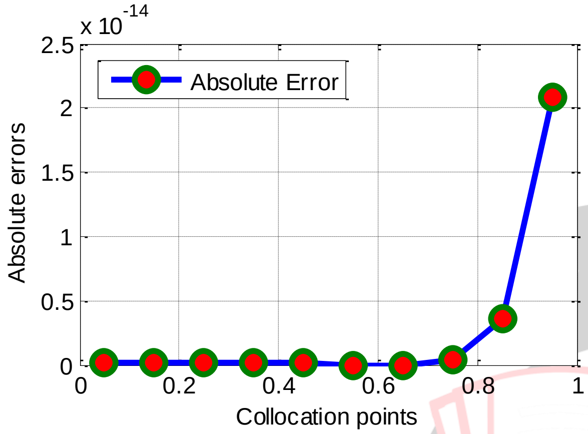


**Figure 4: Absolute errors for Example 3.**

Figure 3 shows the comparison of exact and numerical solutions obtained by Laplace transform method for Example 3 for k=1. Figure 4 shows the absolute errors for Example 3.

**Example 4:** Consider the fourth order ordinary differential equation

$$y^{(iv)} - 10y'' + 9y = 0, \qquad (11)$$

with initial conditions

$$y(0) = 5,\ \ y'(0) = -1,\ \ y''(0) = 21,\ \ y'''(0) = -49.$$

The exact solution of the problem is

$$y(x) = 4e^{x} - e^{-x} + 2e^{-3x}$$

## Adomian Decomposition Method

By using Adomian decomposition method, we get

$$\varphi_0 = 5 - t + \frac{21}{2}t^2 - \frac{49}{6}t^3,$$

$$\varphi_1 = \frac{55}{8}t^4 - \frac{481}{120}t^5 - \frac{21}{80}t^6 + \frac{7}{80}t^7,$$

$$\varphi_2 = \frac{55}{24}t^6 - \frac{481}{504}t^7 - \frac{75}{896}t^8 + \frac{971}{40320}t^9 + \frac{3}{6400}t^{10} - \frac{7}{70400}t^{11},$$

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## Laplace transform method

After taking Laplace transform both sides of (11) and solving , we get

$$(s-1)(s+1)(s+3)\bar{y}(s) = 5s^2 + 14s + 13$$

$$\bar{y}(s) = \frac{5s^2 + 14s + 13}{(s-1)(s+1)(s+3)}$$

After solving, we get

$$\bar{y}(s) = \frac{4}{s-1} - \frac{1}{s+1} + \frac{2}{s+3}$$

Taking inverse Laplace transform, we get

$$y(x) = 4e^{x} - e^{-x} + 2e^{-3x}$$

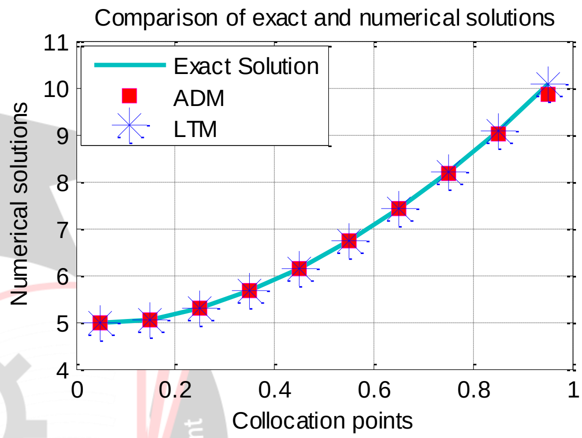


**Figure 5: Comparison of exact and numerical solutions**

Figure 5 shows the comparison of exact and numerical solutions of Example 4.

**Example 5:** Consider the fourth order differential equation

$$y^{(iv)} - 3y'' - 4y = 0 \qquad (12)$$

subject to the initial conditions

$$y(0) = 1, y'(0) = \frac{1}{3}, y''(0) = 0, y'''(0) = 0$$

The exact solution is

$$y(t) = \frac{1}{3}\left[\frac{7}{20}e^{2x} + \frac{1}{4}e^{-x} + \frac{12}{5}cost + \frac{4}{5}sint\right]$$

## Adomian Decomposition Method:

By using Adomian decomposition method, we get

$$\varphi_0 = 1 + \frac{t}{3},$$

$$\varphi_1 = \frac{t^4}{6} + \frac{t^5}{90},$$

$$\varphi_2 = \frac{t^6}{60} + \frac{t^7}{1260} + \frac{t^8}{2520} + \frac{t^9}{68040},$$

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**Laplace Transform Method:**

Taking Laplace transform both side of (12) and after simplification, we get

$$(s^4 - 3s^2 - 4)\bar{y}(s) = \frac{3s^3 + s^2 - 9s - 3}{3},$$

$$\bar{y}(s) = \frac{3s^3 + s^2 - 9s - 3}{3(s^4 - 3s^2 - 4)},$$

$$\bar{y}(s) = \frac{3s^3 + s^2 - 9s - 3}{3(s^2 - 4)(s^2 + 1)},$$

$$\bar{y}(s) = \frac{1}{3}\left[\frac{7}{20(s-2)} + \frac{1}{4(s+1)} + \frac{12s}{5(s^2+1)} + \frac{4}{5(s^2+1)}\right].$$

Taking inverse Laplace transform, we get

$$y(t) = \frac{1}{3}\left[\frac{7}{20}e^{2x} + \frac{1}{4}e^{-x} + \frac{12}{5}cost + \frac{4}{5}sint\right]$$

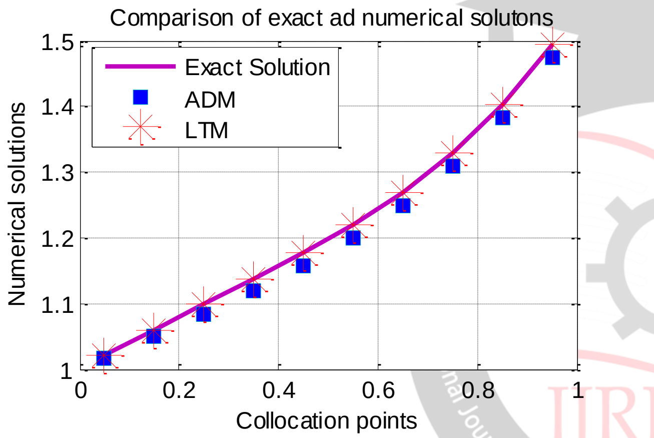


**Figure 6: Comparison of exact and numerical solutions**

Figure 6 shows the comparison of exact and numerical solution of Example 6.

## VII. Conclusion

After discussing some numerical techniques, we have concluded that Laplace transform method is a powerful numerical technique for solving higher order differential equations arising in various applications of science and engineering. Laplace transform method gives more accurate and efficient solutions as compare to other classical numerical techniques.

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