Calculus Instructors' Perceptions of Approximation

Dr. Jay Prakash Tiwari

Professor, Department of Mathematics, Patel College of Science & Technology, Indore, Madhya Pradesh, India. mrjaytiwari31@gmail.com

Abstract - This paper discusses findings from a research study designed to investigate calculus instructors' perceptions of approximation as a central concept and possible unifying thread of the first year calculus. The study also examines the role approximation plays in participants' self-reported instructional practices. A survey was administered to 139 first-year calculus instructors at higher education institutions. Quantitative and qualitative methods were used to analyze the data gathered. Findings from this research will contribute to what is known about the perceptions and teaching practices of calculus instructors regarding the role of approximation in first-year calculus courses. Research-based findings related to the role of approximation ideas in the first-year calculus could have implications for first-year calculus curricula.

Keywords – Calculus,perceptions, Approximation, Curriculum.

I. INTRODUCTION

In recent times the mathematics education community, has become increasingly concerned with issues related to the teaching and learning of calculus. Students enrolled in first-semester calculus courses using traditional instructional approaches made no significant gains in their understanding of the essential calculus concepts measured by CCI pre- and post-testing. On the regional level, the Mathematical Association of America initiated in large-scale study known as *Characteristics of Successful Programmes of College Calculus* to better understand the demographics of students who enroll in calculus and to measure characteristics of calculus courses that may affect student success. Bressoud et al. raised questions about the appropriateness of the calculus curricula in light of their findings that (a) Calculus I is most commonly presented as a first introduction to calculus when more than 25 % of students enrolled at research universities enter with AP[1] Calculus Examination scores of 3 or higher; (b) the calculus curriculum designed for engineering and physical science majors has fundamentally remained the same even though two-thirds of those who enroll in Calculus I today plan to major in other areas; and (c) students reported decreased confidence, enjoyment, and motivation to continue studying mathematics upon completion of their first-semester undergraduate calculus courses. Bressoud et al. also conducted a review of widely adopted Calculus I textbooks coupled with a review of current research literature on student understanding of key calculus concepts and found the research on student learning of key ideas of calculus has had little impact on the conceptual focus of Calculus I curriculum or teaching. In fact, several studies have revealed that many students who receive high grades in Calculus I have weak understanding of the course's key concepts, questions whether the traditional Calculus I curriculum is preparing any students to use ideas of calculus in future courses in mathematics, engineering, or the sciences.

Investigations into the role of approximation in the teaching and learning of calculus have gained momentum over the past decade; however, calls to bring approximation concepts to the fore of first-year calculus curricula are not new. Gordon argued for an early introduction to the approximation of functions, recognizing the local linearity of functions as one of the underlying ideas of the calculus. Further, he identified the approximation of functions as one of the most significant ideas in mathematics and called for it to be more central in first-year calculus courses. Likewise, reform efforts advocated for using approximation as a unifying thread of the first-year calculus curriculum. A *unifying thread* is a concept or theme woven throughout the subject matter which has the potential to bind it together into a cohesive, unified whole. Zorn reported on a working group scientists and mathematicians convened in San Antonio, Texas to discuss the core content of the first-year calculus curriculum.

The theme of approximation, it was agreed, is central to the calculus—what it is and what it does. Two main interpretations of the idea of approximation emerged: approximation in the sense of numerical analysis and approximation as a conceptual idea. The Riemann sum, for instance, can be viewed as either a technique for estimating an integral, or as part of the definition of integral itself. Zorn further articulated the working groups' agreement that, at a minimum, an informal treatment of the basic epsilon-delta idea (i.e., controlling outputs by control- ling inputs) as it occurs in various contexts, such as approximation and error analyses, should be included given its importance to understanding and using the calculus. Reporting on the activity of the same working group in San Antonio, Keynes added that it is important for first-year calculus students to understand the concept of *degrees of accuracy* and think about how good a model can be when modeling physical and statistical data. Keynes also mentioned the group's perception of the importance of asymptotic approximation to a sound understanding of the behavior of functions.

Literature on the use of unifying threads in the teaching and learning of calculus, and mathematics in general, is limited at best. No study has looked at whether using approximation as a unifying thread of the first-year calculus is taking root in the instructional practices of calculus instructors in higher education. The purpose of this study is therefore to examine the following research questions:

1. Do calculus instructors perceive approximation to be important to student under- standing of the first-year calculus?
2. Do calculus instructors report emphasizing approximation as a central concept and- or unifying thread in the first-year calculus?
3. Are there any differences between demographic groups with respect to the approximation ideas they teach in first-year calculus courses?

II. Background Literature

This section outlines relevant literature about the use of unifying threads in the teaching and learning of mathematics, a dynamic view of approximation, and a relatively new body of research suggesting approximation concepts have the potential to help calculus students make sense of key concepts in the first-year calculus curriculum.

Unifying Threads

According to Riggle, the use of curricular unifiers in the teaching and learning of mathematics supports mathematics as a study of structures as opposed to a study of unrelated ideas. For example, Shoenthal has identified Fourier Series as a unifying thread of the Calculus II curriculum B to broaden [students'] appreciation for how interwoven mathematics is in the world around them and to lay the groundwork for future applications of the topic. Framing curricula around unifying threads might address the problem of fragmented learning, which has historically plagued the teaching and learning of mathematics in higher education. Tall addresses the issue of fragmentation in the calculus curriculum: Mathematicians tend to make a typical error when they design an instructional sequence for calculus. The general approach of a mathematician is to try to simplify a complex mathematical topic, by breaking it up in smaller parts, can be ordered in a sequence that is logical from a mathematical point of view. 'From the expert's viewpoint the components may be seen as a part of a whole. But the student may see the pieces as they are presented, in isolation, like separate pieces in a jigsaw puzzle for which no total picture is available. It may be even worse if the student does not realize that there is a big picture. The student may imagine every piece as an isolated picture, which will severely hinder a synthesis. The result may be that the student constructs an image of each individual piece, without ever succeeding in bringing all pieces together in one whole.

To gain a Bbig picture understanding of the first-year calculus, students must build a coherent conception of its underlying mathematical structures. Curricular unifiers can make new math ideas more comprehensible through the construction of connections and relationships to previously studied topics, which support the development of a big picture understanding. Strang notes that calculus instructors are often B very much inside the subject, teaching it but not seeing it. If students are to learn how to Bspeak the language of calculus, instruction must move beyond simply teaching the Bgrammar or rules of the calculus.

A Dynamic View of Approximation

The present study is framed around a *dynamic view* of approximation. According to Ramsey, a *dynamic* view of approximation emphasizes the *process of makingan approximation*, which is essentially a limiting process. BUsed as a verb, 'approximate' means 'to carry or advance near; to cause to approach (to something). An approximation is an act or process and not just a relation. An approximation, thus becomes 'any methodological strategy which is used to generate or interpolate a result due to under-resolved data of deficits of analytic or calculational power.

According to Ramsey, it is commonplace for scientists to hold a *static* view of approximation, which presents approximation as a comparison relation between two structures. In this view, the validity of an approximation is evaluated solely by the magnitude of the error, or by placing a B. limit on the permissible discrepancy between theoretical and experimental values. A dynamic view of approximation considers an expanded set of criteria used to judge the validity of an approximation rather than a simple consideration of the size of the error introduced by an approximation. Three additional criteria for evaluating the validity of an approximation, each internal to the theoretical structure of the approximation, are inherent to a dynamic view: (a) showing that better approximations lead to better predictions or smaller errors (i.e., controllability); (b) proving that the size of the error and the controllability are not the result of chance; and (c) demonstrating good theoretical motivation for the approximation strategy. It is essential, argues Ramsey, to know something about the reliability of the theory and its calculational structure before conclusions can be made about the worth of a theoretical result which matches the experimental result exactly or within a specified range of error.

Leveraging Approximation to Understand Fundamental Concepts in Calculus

A growing body of research suggests approximation ideas can be leveraged to support students' cognitive development of a logical and well-organized collection of connect- ed schemas aligned with key calculus concepts. According to Oehrtman, approximation ideas can be used to help students construct the conceptions needed for formal understanding of limit and limit-related concepts. Oehrtman investigated 120 calculus students' spontaneous reasoning about limit concepts and identified five strong metaphors for limits that served to influence their thinking. The most common was grounded in students' intuitions about approximation and error analyses. According to Oehrtman, approximation metaphors for the limit are not only accessible but powerful given their close resemblance to the correct mathematical structure underlying the limit. Approximation metaphors lay the founda- tion for the eventual development of more formal conceptions of limit and also understanding of the underlying structures of other limit-related concepts of the first- year calculus.

In addition, purposefully chosen approximation problems can help students develop a y-first perspective, which is important in understanding the formal definition of the limit. Swinyard claims calculus students naturally reason about limits from an x-first perspective and, as a result, they are challenged by the formal definition of the limit, which is structured around a y-first perspective. In other words, students find the dependence of delta upon epsilon counterintuitive and struggle with the idea of moving from a condition in the range to a condition in the domain. Crafting problems specifically designed to focus students' attention on the genuine need to reason from a range-first perspective (e.g., finding an approximation with sufficient accuracy for an identified purpose) may help students who otherwise might be unable to attend to the appropriate dependence.

According to Martin, Taylor series are often students' first exposure to function approximation techniques. Students often find it challenging to make sense of Taylor series because it has a complex structure that requires an understanding of many key calculus concepts only some of which include error and error bounds, interval and radius of convergence, and center. A Taylor series is a special case of a power series and is B… a strategy for obtaining better and better approximations to a function at a point by constructing a polynomial whose coefficients are successive derivatives of the function at that point. The Taylor expansion can be thought of as a procedure that takes two arguments, the first being the function that is being approximated and the second is the location of the point about which the Taylor expansion is taking place. According to Yerushalmy and Schwartz there is value in encouraging students to B inspect and analyze the degree to which a given order Taylor expansion is appropriate for a function. Preparing learning activities that cause students to reflect upon the Blocality of the Taylor series expansion can motivate the introduction to other techniques for approximating functions and B can turn approximation into an activity of analyzing, comparing, and even inventing new methods of approximation.

III. Methodology

The following sections outline the development of the survey instrument administered to study participants and the procedures for sampling, data collection and data analysis.

Survey Instrument

A survey instrument was developed to examine approximation-related perceptions and self-reported instructional practices of calculus instructors who have taught first-year calculus courses in higher education. A review of the literature was conducted to establish item stems for the survey. Content validity was established through consultations with six experts in the field. Items stems were added, omitted and refined based upon the feedback of those experts. The survey (see Appendix) includes a series of demographic questions, 20 Likert-scale item stems, an open text box following each Likert-scale item stem to allow participants the option of explaining their rating on the item stem, and two open-response questions.

Procedures for Sampling, Data Collection and Data Analysis

A stratified random sampling method was used to identify the sample for this study. The National Center for Educational Statistics database was used to identify all 2-year and 4-year higher education. A total of 119 institutions were randomly selected through the sampling design. Of those, 37 institutions were excluded from the sample for the following reasons: (a) the institution did not offer calculus courses; (b) the institution had no mathematics department most typically because it was a special-focus institution; or (c) the institution's website did not include publically available mathematics faculty contact information. Despite eliminating 37 institutions for those reasons. A database of 1930 mathematicians was compiled using the name and email address for each mathematician listed on the websites of the selected institutions. All were recruited to participate in the online survey developed for the purpose of this study. Of the 1930 mathematicians recruited, only those who had taught first-year calculus courses were eligible to participate in the study, opening the possibility for a large percentage of ineligible candidates and lower response rate.

Qualtrics, a secure internet-based survey technology provider, was used as the platform to create and distribute the survey. Data were collected over a period of 8 months. Quantitative data were exported to Software Package for the Social Sciences (SPSS) and analyzed using descriptive statistics, t-tests and analysis of variance procedures for statistically comparing the means of the demographic groups of interest. Qualitative data were coded using *a posteriori* categorical content analysis techniques. Members of the research team and trained research assistants isolated dominating themes and defined ranges of themes, indicators for the occurrence of a theme and rules for coding. A total of N=139 calculus instructors, 31 % female and 69 % male, participated in the study. The demographics of the final sample for this study are reported in Table 1.

IV. RESULTS AND DISCUSSION

Discussion of the research findings is organized around four research questions:

1. Do calculus instructors perceive approximation to be important to student under- standing of the first-year calculus?
2. Do calculus instructors report emphasizing approximation as a central concept and- or unifying thread in the first-year calculus?
3. Are there any differences between demographic groups with respect to the approximation ideas they teach in first-year calculus courses?

Area of Specialization			
Applied Mathematics 18%	Math Education 8%	Mathematics 70%	Other 4%

Rank						
Full Professor 29%	Associate Professor 23%	Assistant Professor 19%	Instructor 19%	TA 5%	Emeritus Faculty 2%	Other 3%

Highest Degree Earned		
Ph.D. 74%	Master's 23%	Other 3%

Years of Teaching Experience in Higher Education				
0 – 5 17%	5 – 10 15.25%	10 – 20 24.50%	20 – 30 20%	30+ 23.25%

Years of Experience Teaching Calculus in Higher Education				
0 – 5 22%	5 – 10 21%	10 – 20 24%	20 – 30 18%	30+ 15%

Local / National Awards for Scholarship	
Yes 27%	No 73%

Local / National Awards for Teaching	
Yes 37%	No 63%

Served on Local / National Calculus Committees	
Yes 38%	No 62%

Type of Institution	
Public 66%	Private 34%

Carnegie Classification (Institution)						
Doctorate Degree Granting University 48%	Master's Degree College / University 24%	Baccalaureate Degree College 20%	Associate's Degree College 7%	Special Focus Institution 0%	Tribal College 0%	Other 1%

Calculus Course Taught Most Often[8]									
Standard Calculus I 51%	Honors Calculus I 1%	Calculus I for Math and Science Majors 12%	Standard Calculus II 15%	Honor Calculus II <1%	Calculus II for Math and Science Majors 5%	Calculus with Applications to Social Sciences 2%	Calculus for Business and Economics 6%	Other 1%	Not Identified 6%

Table 1 Summarizes the demographic composition of the study participants

Research Question 1: Do Calculus Instructors Perceive Approximation to Be Important to Student Understanding of the First-Year Calculus?

To answer research question 1, the researchers extracted themes and patterns from participants' written responses to open ended question 21 on the survey instrument. In this study, 89 % of the responding participants agree that approximation ideas are important to student understanding of the first-year calculus. There is essentially no topic in the application of calculus in which approximation does not play a central role. The theory of calculus is really a theory of approximation. The best way to understand the idea of a limit is in terms of approximation and acceptable error.

Several themes emerged from the data shedding light on reasons participants perceive approximation to be important to student understanding of the first-year calculus, including: (a) approximation is a primary building block of the calculus, (b) approximation is foundational to understanding the fundamental concepts in calculus, (c) approximation can motivate the study of calculus topics and- or make calculus topics more meaningful to students, (d) approximation has relevance to real world or applied problems, which are rarely exact, (e) approximation is a critical component of the knowledge base of science, technology, engineering, and mathematics [STEM] majors, and (f) approximation ideas have historical significance to the development of the calculus. Participant R_1S7 wrote, B[Approximation] is the foundation of the limit concept which, in turn, is the unifying concept of differential and integral calculus and of infinite series. Why bother to discuss infinite series at all if not for the purpose of obtaining algebraic or trigonometric approximations to transcendental functions?

Research Question 2: Do Calculus Instructors Report Emphasizing Approximation as a Central Concept and- or Unifying Thread in the First-Year Calculus?

To answer research question 2, the researchers examined data associated with Item Stems 1 and 2 on the survey instrument. Analyses of the data reveal that 51 % of the responding study participants BAgree or BStrongly Agree that they present approximation as a central concept in their own teaching of the first-year calculus. Fewer (40 %) BAgree or BStrongly Agree^ that they use approximation as a unifier of the first-year calculus curriculum in their own teaching (see Fig. 2). Five themes emerged that explain calculus instructors' emphasis on approximation as a central concept and-or unifying thread in their teaching a) approximation can illuminate reasons for learning calculus and-or help students see the Bbig picture; b) most interesting functions are not elementary functions and approximations are useful in dealing with those situations; c) approximation ideas facilitate understanding of fundamental concepts in the first-year calculus and, therefore, reduce the likelihood that calculus will

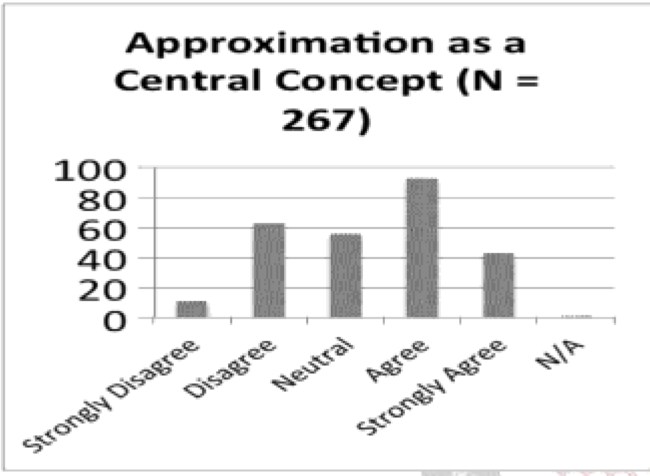

Fig. 1 Participants' agreement that approximation is emphasized as a central concept in their own first-year calculus course(s)

degenerate into a study of rote and meaningless computation; (d) linear approximation is the foundation of differential calculus; and (e) participants with a background in applied mathematics or numerical analysis acknowledged their specialization as a factor in the emphasis they place on approximation in their calculus courses.

A number of themes emerged among those participants who do not emphasize approximation as a central concept and-or unifying thread in their first-year calculus courses. First, the most common theme was that approximation is not germane to enough topics in the first-year calculus to warrant excessive emphasis. Stated Partici- pant R_0pl, BI use approximation to motivate the precise definitions of the derivative of a function at a point or the value of a definite integral, but I would not claim that approximation is the central concept that I wish to convey.^ Likewise, some

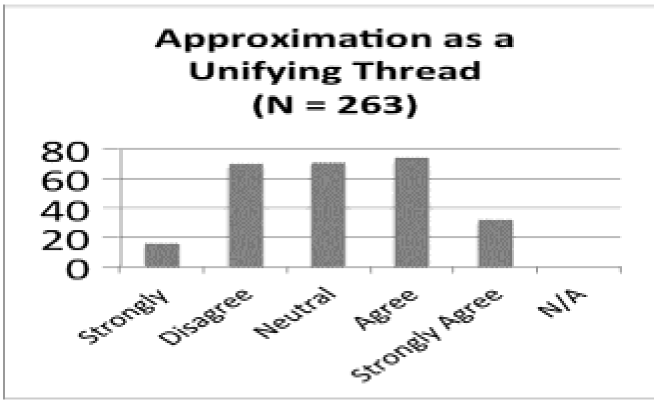

Fig. 2 Participants' agreement that approximation is used as a unifying thread in their own first-year calculus course(s)

participants viewed approximation as an application of calculus and could not justify more than a peripheral emphasis on it in their teaching of first-year calculus.

Second, constraints such as overcrowded calculus syllabi, limited technology access or math department-imposed technology bans, little freedom to make curricular decisions, and ill-prepared students fail to afford some calculus instructors options for presenting approximation as central or unifying in their first-year calculus courses. The response of Participant R_cFM typified that concern: BYes, is very important; however, due to time constraints and student preparation, namely, I get bogged down on explaining a lot of the material because my students are not generally strong. I am not able to do justice to the role of approximation to the extent that I'd like. Those participants were optimistic that students studying STEM disciplines would have adequate exposure to approximation ideas in subsequent numerical analysis coursework.

Third, some participants reported presenting other concepts (e.g., limit, study of change) as central and-or unifying in their first-year calculus courses. According to Participant R_0Bc, BGenerally, I follow whatever is the suggested text. Approximation is right there in the notion of a limiting process, which of course, is

central to the definitions of derivatives and integrals. But probably I would say the notion of limit, the thing that ultimately 'beats all approximations' is really the central concept.

Finally, some participants expressed concerns surrounding the use of technology in the teaching and learning of calculus. According to Participant R_9GH, approximation: is an important application. However, after many years of trying many different approaches, including using technology, I have found too much technology detracts from a calculus course and many students end up with a misunderstanding of the importance of mathematics and, in particular, calculus.

Similarly, Participant R_d5n observed, BApproximation arises in limits, differentia- tion, integrals, differentials, and error estimates, but I try to be careful because too many students will then use a calculator to find answers rather than their own brain.

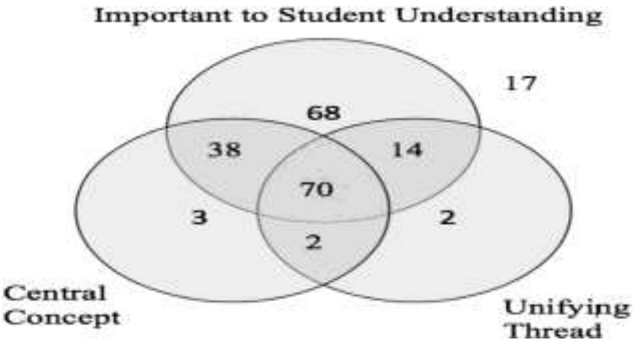

Fig. 3 Graph depicting participants' perceptions of approximation

The diagram in Fig. 3 summarizes the perceptions of responding participants regarding the role of approximation in their own first-year calculus courses. For instance, 39 participants responded that they *do* perceive approximation to be important to student understanding of the first-year calculus, *do* emphasize approximation as a central concept, *do* use approximation as a curricular unifier in their own teaching; 37 participants responded that they *do* perceive approximation to be important to student understanding of the first-year calculus, *do not* emphasize approximation as a central concept, *do not* emphasize approximation as a unifying thread in their own teaching; 37 participants responded that they *do* perceive approximation to be important to student understanding of the first-year calculus, *do* emphasize approximation as a central concept, *do not* use approximation as a curricular unifier in their own teaching.

Research Question 3: Which Approximation Ideas Do Calculus Instructors Believe are B Worthwhile to Address in First-Year Calculus Courses?

To answer research question 3, the researchers examined data associated with Items Stems 3–20 on the survey instrument. The findings in this section are organized around the topics of error analyses, derivative concepts, functions, series and definite integrals.

Error Analyses

Findings related to error analyses focused on tolerance, estimating error, acceptable levels of error, and discriminating between approximation techniques. BIf the errors are not tolerable, approximations are useless^ stated Participant R_e4w. Fifty percent of the responding participants reported stressing the importance of knowing how good an approximation is in their first-year calculus courses, though more so in second-semester calculus than first-semester courses. Participant R_89a captures participants' views surrounding the significance of context in determining how good an approximation is We often, several times in the semester, discuss whether an error of, say, 0.001 is 'good.' Some students are aware and the others become aware that the scale and setting make all the difference. Relative error is the indicator we want. A measure of 0.001 m is useless compared to the size of a hydrogen atom.

Fifty seven percent of responding participants BAgree or BStrongly Agree that they discuss methods for calculating or estimating the error in an approximation, particularly when discussing Taylor series, differentials, linearization, approximate values of linear functions, and numerical methods for estimating definite integrals. Numerical integration and Taylor series were identified as topics in which students can readily discriminate between various approximation techniques, though only 34 % of participants reported including those kinds of investigations in their first-year calculus courses. As previously noted, the focus for some was simply on *discussion* around the importance of knowing how good an approximation is and not on actual error calculations.

Forty-four percent of responding study participants BAgree or BStrongly Agree that they discuss the notion of acceptable levels of error in an approximation. Participant R_6Rw responded I usually point out that this is an important issue, but it is secondary to under- standing what they are approximating and getting an approximation to start with. If students don't know what a derivative is, there's not much point in worrying about how good a specific estimate of it is. Sometimes though, the issue of accuracy of an estimate can help inform their understanding of a procedure, such as the rectangle rule for estimating a definite integral.

Participants reported several reasons for not devoting attention to error analyses in their first-year calculus courses. Lack of time again surfaced as a major constraint, as did instructor perceptions of weak mathematical backgrounds among first-year calculus students. Participants reported that first-year calculus students often struggle to grasp the concept of tolerance, quantify the error in an approximation, or appreciate the subtle notion of *acceptable levels of error*. If discussed at all, *acceptable levels of error* might be demonstrated in an example or two (e.g., positioning a robot arm within some specified tolerance of a target location by determining how accurately the hydraulic pressure that activates the arm must be controlled). Other reasons for deemphasizing error analyses in first-year calculus courses included lack of alignment with (non-honors) course learning goals, difficulty assessing student understanding, and student disinterest in error analyses concepts. Participant R_cBd responded, BThe kids in my class are typically not interested in such fine points. Thus, when I try to explain such things, they usually become distracted knowing I cannot test on such material. I would like to do more, but it doesn't really fit into a freshman calculus course.

Derivative Concepts

The findings related to derivative concepts focused on approximating the slope of the tangent line and bounding the error in an approximation of the slope of the tangent line. Not surprisingly, 95 % of responding participants BAgree^ or BStrongly Agree^ that they show students how to approximate the slope of the tangent line using secants.

Participants suggested that this approximation is an accessible concept for students that is easily demonstrated using technology and represents one way to introduce the idea of the derivative. Given the results related to error analyses reported in Section BError Analyses, it is also not surprising that significantly fewer responding participants (69 %) reported explaining to their first-year calculus students that the error in approximating the slope of a tangent line can be made smaller than any predetermined bound. A number of participants reported that, if they do discuss the error in the slope of the tangent line, it is non-rigorous and connected to limit concepts. Participant R_9mH responded, BI present this idea to my students, but I do not ask them to demonstrate this fact themselves. What I emphasize is that in the limit, this approximation becomes exact. Participant R_9mH responded that the question of precision is properly handled by discussion of the second derivative: B…This I consider to be part of the more general Taylor polynomial discussion that is had in Calculus II, where the 'next' derivative is proven to relate to the Taylor error.

Functions

The findings related to function concepts focused on linearization and approximation of functions. There was strong agreement among responding participants (83 %) that they discuss linearization techniques in their first-year calculus courses. Linearization was described as Bthe essence of calculus^ by Participant R_cBd and as B…a fundamental theme of the chapters on differentiation^ by Participant R_81v. Clearly perceived as a central idea of the calculus, linear approximation was even described by some participants as a possible Bbridge concept, in the first year calculus. For instance, Participant R_cTs wrote, BThis helps set the stage for Taylor polynomials in the next course which, from my experience teaching it, is often a hard idea for students to grasp without connecting it first to the linearization done in first-semester calculus. Likewise, Participant R_0Bc responded, BYes, linear approximation is central, and it is the part that generalizes beautifully in multivariable calculus, linear algebra, differential geometry, and beyond. While only 44 % of responding participants BAgree or BStrongly Agree that function approximation is emphasized as a main theme in their first-year calculus courses, analysis of written comments from the complementary 56 % of participants indicated that they generally *do* address approximation of functions intheir first-year calculus courses and view it as important, but would simply not classify it as a Bmain theme.

Series

Study findings related to series focused on motivating the study of power series, approximating the values of complicated functions using power series, bounding error, estimating the error term in a Taylor polynomial, number of terms in a Taylor polynomial, and approximating a finite or infinite sum using power series.

Seventy-two percent of responding participants reported that they discuss reasons for studying power series in their first-year (but mostly second-semester) calculus courses. According to Participant R_6Rw, BTo the extent that one now covers power series, or anything for that matter, one has to motivate them. Since power series seem hard for students, [they] need more motivation for working on them than for some other topics. Seventy percent of participants BAgree or BStrongly Agree that they demonstrate how power series can be used to approximate the values of complicated functions and believe students to like those demonstrations. Participant R_cTs reported. This is my students' favorite class as I'm able to show via Mathematica examples of how well the polynomials approximate functions. There is also a small demo they can view at home—a game where they try to avoid a projectile that is aimed using increasing orders of polynomials, with readouts showing their position, speed, and acceleration that are used to select the angle of fire.

Seventy-two percent of responding participants BAgree or BStrongly Agree that they emphasize that the partial sum of a power series represents an approximation of a function at a point and, within the interval of convergence, this approximation can be made as accurate as possible by increasing the number of terms in the partial sum.

Fifty-eight percent of responding participants BAgree or BStrongly Agree that they demonstrate how to estimate the error term in order to evaluate how good a Taylor Polynomial approximation is, while 53 % demonstrate how to use the error term for a Taylor Polynomial to determine how many terms of a power series are sufficient to guarantee that an approximation has a given accuracy. Participant R_89a comments, BIn my opinion, there's not much point unless you can bound the error. I also teach this in Calculus I, where I explain that it is plausible that the second derivative controls the error of a linear approximation; then we use (without formal proof) that bound.^ Written comments suggested time constraints and weak student backgrounds impede some calculus instructors from devoting attention to demonstrations of using the error term to evaluation how good a Taylor Polynomial approximation is or to determine the number of terms needed to guarantee a Taylor polynomial approximation has a given accuracy. While some participants indicated that using definite integrals to approximate a finite or infinite sum is a year 2 calculus topic at their institutions, 69 % of responding participants BAgree or BStrongly Agree that they show how to approximate a finite or infinite sum using definite integrals in their first-year calculus course.

Definite Integral

Study findings related to definite integrals focused on techniques for approximating definite integrals, Riemann sums, error in approximating definite integrals, and examples of definite integrals best approximated numerically. Eighty-five percent of the responding participants in this study discuss techniques for approximating definite integrals in their first-year calculus courses and 72 % share examples of definite integrals that are best approximated numerically. Participant R_8Ar reported, BI think this is essential if students are to understand the integral as something more than the result of anti differentiation. Specific approximation techniques identified included rectangular Riemann approximations (left, right, midpoint), trapezoidal Riemann approximations, and Simpson's Rule; although, calculus pro- grams have omitted the latter two techniques according to some participants. There was also strong agreement (93 %) among participants for using Riemann sums to discuss estimating the value of a definite integral; however, they cautioned that students can find Riemann sums difficult, unimportant, or even—in the case of those who have already had exposure to calculus in high school—burdensome.

Research Question 4: Are There Any Differences Between Demographic Groups with Respect to the Approximation Ideas Taught in First-year Calculus Courses

T-test and ANOVA procedures were used to identify significant differences between demographics groups on item stems 1–20. Analysis of survey data showed a number of significant differences between the group of participants who reported having served on calculus committees at the local and-or national level (N=103) and the group of participants who reported never having done so (N=170). Table 2 shows the item stems on the survey for which there were significant differences between the two groups:

All participants who had served on local and-or national calculus committees agreed more strongly with the item stems identified in Table 3. They agreed more strongly that they (a) emphasize approximation as a central concept. (b) use approximation as a unifying thread, (c) discuss linearization, and (d) share examples of definite integrals that are best approximated numerically. They also agreed more strongly with item stems 8 and 19, which involve error analyses.

V. Conclusions and Implications

This baseline study provides both quantitative and qualitative findings on whether the responding first-year calculus instructors view approximation to be a central concept and-or a unifying thread of first year calculus and if their perceptions about approximation ideas are reflected in their reported instructional practices. To this point, no study has directly investigated what first-year calculus instructors perceive to be the role of approximation in the teaching and learning of the first-year calculus. The large majority (89 %) of study participants agreed that students' understanding of approximation is important to their understanding of the first-year calculus. Those participants further elaborated that approximation is a primary building block of the calculus and is foundational to the development of an understanding of many of the key concepts in the first-year calculus. This finding alone underscores the need to probe deeper to understand how calculus instructors frame their courses to include approximation concepts and, more specifically, which approximation ideas calculus instructors per- ceive to be worth addressing in their actual teaching of the first-year calculus. While 89 % of participants reported that they do view approximation as important to student understanding of the first-year calculus, significantly fewer are translating that view into an instructional approach that presents approximation as a central concept (51 %) or unifying thread (40 %). Participants reported the following four primary reasons for such an incongruity:

First, some participants of the present study reported that they do not perceive approximation to be a curricular unifier of the first-year calculus inasmuch as it is not germane to a sufficient number of topics in the first-year calculus and is therefore better presented as an application of calculus.

Second, participants identified a number of impediments to emphasizing approximation as a central concept and-or unifying thread in their own teaching of first-year calculus courses: (a) many participants cited an already overcrowded calculus syllabus with no room or time to integrate approximation-related course learning goals; (b) others expressed that they had limited freedom to make curricular decisions in mathematics departments that often compelled the adoption of common first-year calculus syllabi and-or textbooks lacking an emphasis on approximation ideas or other more conceptually-oriented topics.

Third, some participants reported a preference to emphasize other concepts or themes to unify the first-year calculus curriculum (e.g., limit, change) or to simply focus on the techniques of calculus rather than approximation ideas.

Fourth, concerns were raised by some participants related to the role and availability of technology in the teaching and learning of first-year calculus as reasons for not emphasizing approximation as central or unifying in the first-year calculus.

To teach approximation as a unifying thread or not, that really is the big question. There is no one-curriculum-fits-all in first year calculus today. Mathematics departments omit or emphasize topics differently to design a first-year calculus program that reflects the purpose of learning calculus, especially given present-day techno- logical innovations. Notwithstanding, research referenced in this paper is suggesting that approximation ideas are foundational to the *Calculus* and form part of its connective tissue. Perhaps participants in this study expressed it best: To learn calculus without understanding approximation ideas is to learn calculus without meaning.

REFERENCES

[1] Asiala, M., Cottrill, J., Dubinsky, E., & Schwingendorf, K. E. (1997). The development of students' graphical understanding of the derivative. The Journal of Mathematical Behavior, 16(4), 399–431.

[2] Baroody, A. J., Feil, Y., & Johnson, A. R. (2007). An alternative reconceptualization of procedural and conceptual knowledge. Journal for Research in Mathematics Education, 38(2), 115–131.

[3] Berry, J. S., & Nyman, M. A. (2003). Promoting students' graphical understanding of the calculus. The Journal of Mathematical Behavior, 22, 481–497.

[4] Bressoud, D. M., Carlson, M. P., Mesa, V., & Rasmussen, C. (2013). The calculus student: insights from the Mathematical Association of America national study. International Journal of Mathematical Education in Science and Technology, 44(5), 685–698. doi:10.1080/0020739X.2013.798874.

[5] Chai, J., Friedler, L. M., Wolff, E. F., Li, J., & Rhea, K. (2015). A cross-national study of calculus. International Journal of Mathematical Education in Science and Technology, 46(4), 481–494. doi:10. 1080/0020739X.2014.990531.

[6] College Board, The. (2015). Annual AP program participation. Retrieved from http://research.collegeboard. org/programs/ap/data/participation/ap-2014.

[7] Czocher, J. A., Tague, J., & Baker, G. (2013a). Coherence from calculus to differential equations. In S. Brown,

[8] G. Karakok, K. H. Roh, & M. Oehrtman (Eds.), Proceedings of the 16th Annual Conference on Research in Undergraduate Mathematics Education: Vol. 2, Denver, CO.

[9] Czocher, J. A., Tague, J., & Baker, G. (2013b). Where does calculus go? An investigation of how calculus ideas are used in later coursework. International Journal of Mathematical Education in Science and Technology, 44(5), 673–684. doi:10.1080/0020739X.2013.780215.

[10] Dorier, J. (1995). Meta level in the teaching of unifying and generalizing concepts in mathematics.Educational Studies in Mathematics, 29, 175–197.

[11] Dray, T., & Manogue, C. (2010). Putting differentials back into calculus. The College Mathematics Journal, 41(2), 90–100.

[12] Epstein, J. (2013). The calculus concept inventory: Measurement of the effect of teaching methodology in mathematics. Notices of the AMS, 60(8). Retrieved from: http://www.ams.org/notices/201308/rnoti-p1018.pdf.

[13] Ferrini-Mundy, J., & Graham, K. (1994). Research in calculus learning: Understanding of limits, derivatives, and integrals. In J. Kaput & E. Dubinsky (Eds.), Research Issues in Undergraduate Mathematics Learning, (MAA Notes #33, pp. 29–45). Washington, D.C.: Mathematical Association of America.

[14] Galbraith, P., & Haines, C. (2000). Conceptual mis(understandings) of beginning undergraduates. International Journal of Mathematical Education in Science and Technology, 31(5), 651–678.

[15] Gordon, S. P. (1993). Calculus must evolve. Primus, 3(1), 11–18.

[16] Gordon, S. P. (2011). A classroom note on: building on errors in numerical integration. Mathematics and Computer Education, 45(2), 148–156.

[17] Gordon, S. P. (2012). Some surprising errors in numerical differentiation. Primus, 22(6), 437–450.

[18] Gravemeijer, K., & Doorman, M. (1999). Context problems in realistic mathematics education: a calculus course as an example. Educational Studies in Mathematics, 39, 111–129.

[19] Hathaway, R. J. (2008). A simple acronym for doing calculus: CAL. Primus, 18(6), 542–545.

[20] Heim, B., Rupp, F., Viet, N., Stockhausen, P. V., Gallenkämper, J., & Kreuzer, J. (2015). Driving student- centered calculus: results of a comprehensive case study for Kaizen learning in the Sultanate of Oman. International Journal of Mathematical Education in Science and Technology, 46(3), 354–369. doi:10. 1080/0020739X.2014.979897.

[21] Hiebert, J., & Lefevre, P. (1987). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), Conceptual and procedural knowledge: The case of mathematics (pp. 1–27). Hillsdale: Erlbaum.

[22] Hilbert, S., Schwartz, D. D., Seltzer, S., Maceli, J., & Robinson, E. (2010). Calculus: An active approach with projects. Washington, D.C.: MAA.

[23] Jungić, V., & Mulholland, J. (2014). Is close enough good enough? Reflections on the use of technology in teaching calculus. International Journal of Mathematical Education in Science and Technology, 45(7), 1075–1084. doi:10.1080/0020739X.2014.892166.

[24] Kannemeyer, L. (2005). Reference framework for describing and assessing students' understanding in the first year calculus. International Journal of Mathematical Education in Science and Technology, 36(2–3), 271–287.

[25] Keynes, H. B. (1991). The calculus reform movement: some views from the outside. Primus, 1(4), 359–367.

[26] Klopfenstein, K., & Thomas, K. M. (2009). The link between Advanced Placement experience and early college success. Southern Economic Journal, 75(3), 873–891.

[27] Kortendick, O. & Fischer, M. (1996). Comparative classifications of ethnographic data: Constructing the outline of cultural materials. CSAC Studies in Anthropology, 11. Retrieved from http://lucy.ukc.ac.uk/ CSACSIA/Vol14/Papers/Kortendick/OCMPaper/OCMPaper_1.html #H1.

[28] Lang, X. (1999). CAI and the reform of mathematics education in China. International Journal of Mathematical Education in Science and Technology, 30(3), 399–404.

[29] Mahir, N. (2009). Conceptual and procedural performance of undergraduate students in integration. International Journal of Mathematical Education in Science and Technology, 40(2), 201–211.

