International Journal for Research in Engineering Application & Management (IJREAM)
ISSN : 2454-9150 Vol-05, Issue-02, May 2019

Coupled fixed point theorem for generalized F-contraction in complete Complex valued metric space

[1]Neelu Rajput, [2]Naval Singh
[1]H.No.2, NH bungalows, DK-3, Danish Kunj Kolar Road, Bhopal-462042, India
[2]Department of Mathematics, Govt.Dr.Shyama Prasad Mukherjee science and commerce college (Benazeer) Bhopal, India

Abstract **Wardowski introduced a new concept of contraction and proved a fixed point theorem which generalizes Banach contraction principle. Following this direction of research, in this paper, we prove common Coupled fixed point theorem which satisfy an F- contractive condition for self mapping in complete complex valued metric space.**

MSC: 47H10, 54H25.

Key words: Complex Valued metric space, Coupled fixed point, F- contraction.

1. Introduction and preliminaries:

The fixed point theory is very important and useful in mathematics because of its application in various areas such as variation and linear inequalities approximation theory physics and computer science. the Banach contraction principle [3] is very popular and effective tool in solving existing literature of fixed point theory contain a great no of generalizations of Banach contraction principle by using different form of contraction condition in various space but majority of such generalization are obtained by improving underlying contraction condition which also includes contraction condition described by rational expressions. In 2011 Azam et al [2] introduce the notation of complex valued metric space and established some fixed point results for pair of mapping for contraction condition satisfying a rational expression

$Let\ C\ be\ the\ set\ of\ complex\ numbers\ and\ let\ z_1, z_2 \in C.$

$Define\ a\ partial\ order \precsim on\ C\ as\ follows: z_1 \precsim z_2$

$if\ and\ only\ if\ Re(z_1) \le Re(z_2),\ Im(z_1) \le Im(z_2).$

$It\ follows\ that:\ z_1 \precsim z_2\ if\ one\ of\ the\ following$

$conditions\ is\ satisfied:$

(1) $Re(z_1) = Re(z_2), \mathrm{Im}(z_1) < \mathrm{Im}(z_2),$

(2) $Re(z_1) < Re(z_2), \mathrm{Im}(z_1) = \mathrm{Im}(z_2),$

(3) $Re(z_1) = Re(z_2), \mathrm{Im}(z_1) = \mathrm{Im}(z_2).$

$In\ particular, we\ will\ write\ z_1 \precnsim z_2$ if $z_1 \neq z_2\ and\ one\ of$

$(1), (2)\ and\ (3)\ is\ satisfied\ and\ we\ will\ write\ z_1 \prec z_2\ if\ only\ (3)\ is\ satisfied.$

***Definition* 1.1**: [2] $Let\ X\ be\ a\ nonempty\ set. Suppose\ that\ the\ mapping$

$d : X \times X \to C \quad Satisfies:$

(a) $0 \precsim d(x,y)\ for\ all\ x,y \in X\ and\ d(x,y) = 0\ if\ and\ only\ if\ x = y$

(b) $d(x,y) = d(y,x)\ for\ all\ x,y \in X$

(c) $d(x,y) \precsim d(x,z) + d(z,y)\ for\ all\ x,y,z \in X.$

$Then\ d\ is\ called\ a\ complex\ valued\ metric\ on\ X\ and\ (X,d)\ is\ called\ a\ complex\ valued\ metric\ space.$

***Definition*1.2**: [2] $Let\ \{x_n\}\ be\ a\ sequence\ in\ X\ and\ x \in X. If\ for\ every\ c \in C, with\ 0 < c$

$there\ is\ n_0 \in N\ such\ that\ for\ all\ n > n_0, d(x_n, x) < c,$ then x $is\ called\ the\ limit\ of$

$\{x_n\}\ and\ we\ write\ \lim_{n\to\infty} x_n = x\ or\ x_n \to x\ as\ n \to \infty.$

***Definition* 1.3**: [2] $If\ every\ Cauchy\ sequence\ is\ convergent\ in\ (X,d), then\ (X,d)\ is\ called\ a$

$Complete\ Complex\ valued\ metric\ space.$

***Lemma* 1.4**: [2]$Let\ (X,d) be\ a\ complex\ valued$

$metric\ space\ and\ \{x_n\}\ a\ sequence\ in\ X,$
$Then\ \{x_n\}\ converges\ to\ x\ if\ and\ only\ if\ |d(x_n,x)| \to 0\ as\ n \to \infty.$

***Lemma* 1.5**: [2]$Let\ (X,d)\ be\ a\ complex\ valued$

$metric\ space\ and\ \{x_n\}\ asequence\ in\ X. Then$

$\{x_n\}\ is\ a\ Cauchy\ sequence\ if\ and\ only\ if\ |d(x_n, x_{n+m})| \to 0\ as\ n \to \infty\ where\ m, n \in N.$

 DOI : 10.35291/2454-9150.2019.0048

Definition 1.6:[6] An element (x, y) Є X× *X* is called a coupled fixed point of T: X × X → X if x =T(x, y) , y =T(Y, x)

Definition 1.7:[6] An element (x,y) Є X× *X* is called a coupled coincidence point of , T: X × X → X if S(x,y) =T (x,y) , S(y,x) =T(y,x)

Definition 1.8:[6] An element (x,y) Є X× *X* is called a common coupled fixed point of S, T: X × X → X if x = S(x,y) =T (x,y), y = S(y,x) = T(y,x)

In 2012,Wardowski [11] introduced a new concept of contraction, and he proved a fixed point theorem which generalizes the Banach contraction principle.Later on,Wardowski and Van Dung [10] gave the idea of *F*-weak contraction and proved a theorem concerning *F*-weak contraction.Afterwards, Abbas *et al.* [1] further generalized the concept of *F*-contractionand proved certain fixed point results. Batra *et al.* [4,5] extended the concept of *F*-contraction on graphs and altered distances. They proved some fixed point and coincidence point results by illustrating them with some examples. Recently, Cosentino and Vetro [7] followed the approach of *F*-contraction and obtained some fixed point theorems of Hardy-Rogers-type for self-mappings in complete metric spaces and complete ordered metric spaces. Then Sgroi and Vetro [9] extended this Hardy-Rogers-type fixed point result for multivalued mappings.

First we recall the concept of F- contraction ,which was introduced by Wardowski [11]

Let $\mathcal{F}$ be the set of all functions F: $\mathbb{R}^+ \to \mathbb{R}$ satisfying the following conditions:

(F1) *F* is strictly increasing;

(F2) for all sequence { α_n } ⊆ $\mathbb{R}^+$, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$

(F3) there exists 0< k < 1 such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$

Definition 1.9 :[11] Let (*X, d*) be a metric space. A mapping *T: X →X* is said to be an F-contraction if there exist τ Є $\mathbb{R}^+$and a function F Є $\mathcal{F}$ such that for ∀ x,y Є X,

$$d(Tx,Ty) > 0 \Longrightarrow \tau + F(d(Tx,Ty)) \le F(d(x,y)) \quad (1)$$

Obviously F (α) = ln (α) and F (α) = $\frac{-1}{\sqrt{\alpha}}$,for $\alpha > 0$ satisfy in the above condition .

Remark 1.10: From (F1) and (1), it is easy to conclude that every F contraction is necessarily continuous.

Example1.11: [17] Let F: $\mathbb{R}^+ \to \mathbb{R}$ be defined by $F(\alpha) = \ln\alpha$. It is clear that *F* satisfies (F1)-

(F3) for any *k* Є (0, 1). Each mapping *T: X →X* satisfying (1) is an *F*-contraction such that

$d(Tx, Ty) \le e^{-\tau} d(x, y)$ for all *x, y* Є *X, Tx ≠ Ty*.

It is clear that for *x, y* Є *X* such that *Tx* = *Ty*, the inequality $d(Tx, Ty) \le e^{-\tau} d(x, y)$ also holds, i.e.Banach contraction principle.

Wardowski [11] stated a modified version of the Banach contraction principle as follows.

Theorem 1.12 [11]: *Let* (*X, d*) *be a complete metric space and let T: X→X be an F-contraction.*

Then T has a unique fixed point z Є *X and for every x* ∈ *X the sequence* $\{T^n x\}$ $n \in \mathbb{N}$

converges to z.

2. Main Result:

Proposition2.1: Let (X,d) be a complete complex valued metric space and T : X → X be a mapping .let $x_0 \in X$,and defined the sequence $\{x_n\}$ by $x_{2n+1} = Sx_{2n}$ $x_{2n+2} = Tx_{2n+1}$ for all $n \ge 0$.

Assume that there exists a mapping λ: X × X → [0,1) such that $\lambda(TSx, y) \le \lambda(x, y)$and $\lambda(x, STy) \le \lambda(x, y)$ for all x,y ∈ X ,then
$\lambda(x_{2n}, y) \le \lambda(x_0, y)$ *and* $\lambda(x, x_{2n+1}) \le \lambda(x, x_1)$

Proof: let all x,y ∈ X for all $n \ge 0$ then we have

$$\lambda(x_{2n}, y) = \lambda(TSx_{2n-1}, y) \le \lambda(x_{2n-2}, y) = \lambda(TSx_{2n-4}, y) \le \cdots \le \lambda(x_0, y).$$

Similarly we have

$$\lambda(x, x_{2n+1}) = \lambda(x, TSx_{2n-1}) \le \lambda(x, x_{2n-1}) = \lambda(x, TSx_{2n-3}) \le \cdots \le \lambda(x, x_1)$$

Now we prove our main theorem.

***Theorem* 2.2**: Let (X,d) be a complete complex valued metric space and T : X × X → X be a mapping. if there exists $\tau > 0$ and mapping λ, μ, δ : X × X → [0,1) such that for all x,y ∈ X :

(i) λ (T(x,y)) ≤ λ (x,y) and μ (T(x,y)) ≤ μ (x,y) , δ(T(x,y)) ≤ δ(x,y)

(ii) ($\lambda + \mu + \delta$) (x, y) < 1

(iii) d(T(x,y),T(u,v)) > 0 ⟹ τ + F(d (T(x,y),T(u,v)) ≲ F(λ (x,y) d(T(x,y),x) + μ (x,y) d(T(u,v),u)

$$+ \ \delta(x,y) \frac{d(T(x,y),u),d(T(u,v),x)+d(T(x,y),x)d(T(u,v),u)}{1+d(x,u)})$$

Where F Є $\mathcal{F}$ Then T has a unique coupled fixed point.

Proof: Choose x_0 , y_0 ∈ X and set x_1 = T (x_0, y_0), y_1 =T (y_0, x_0),

...

x_{2n+1} =T (x_{2n}, y_{2n}), y_{2n+1} =T (y_{2n}, x_{2n}).

Now from proposition 2.1

 DOI : 10.35291/2454-9150.2019.0048

τ + F (d (x_{2n}, x_{2n+1})) = τ + F (d (T (x_{2n-1}, y_{2n-1}), d (T (x_{2n}, y_{2n}))

$\lesssim$ F (λ (x_{2n-1},y_{2n-1}) d (T (x_{2n-1},y_{2n-1}),x_{2n-1}) + μ (x_{2n-1},y_{2n-1}) d (T (x_{2n},y_n),x_{2n})

+ δ (x_{2n-1}, y_{2n-1}) $\frac{d(T(x_{2n-1},y_{2n-1}),x_{2n}),d(T(x_{2n},y_{2n}),x_{2n-1})+d(T(x_{2n-1},y_{2n-1}),x_{2n-1})\,d(T(x_{2n},y_{2n}),x_{2n})}{1+d(x_{2n-1},x_{2n})}$)

= F (λ (x_{2n-1},y_{2n-1}) d (x_{2n},x_{2n-1}) + μ (x_{2n-1},y_{2n-1}) d (x_{2n+1},x_{2n})

+ δ (x_{2n-1}, y_{2n-1}) $\frac{d(x_{2n},x_{2n}),d(x_{2n+1},x_{2n-1})+d(x_{2n},x_{2n-1})d(x_{2n+1},x_{2n})}{1+d(x_{2n-1},x_{2n})}$)

$\lesssim$ F (λ (T(x_{2n-1},y_{2n-1})) d (x_{2n},x_{2n-1}) + μ T(x_{2n-1},y_{2n-1}) d (x_{2n+1},x_{2n})

+ δ T(x_{2n-1}, y_{2n-1}) $\frac{d(x_{2n},x_{2n}),d(x_{2n+1},x_{2n-1})+d(x_{2n},x_{2n-1})d(x_{2n+1},x_{2n})}{1+d(x_{2n-1},x_{2n})}$)

= F (λ ((x_{2n-2},y_{n-2})) d (x_{2n},x_{2n-1}) + μ (x_{2n-2},y_{2n-2}) d (x_{2n+1},x_{2n})

+ δ (x_{2n-2}, y_{2n-2}) $\frac{d(x_{2n},x_{2n}),d(x_{2n+1},x_{2n-1})+d(x_{2n},x_{2n-1})d(x_{2n+1},x_{2n})}{1+d(x_{2n-1},x_{2n})}$)

$\lesssim$ F (λ ((x_0,y_0)) d (x_{2n},x_{2n-1}) + μ (x_0,y_0) d (x_{2n+1},x_{2n})

+ δ (x_0, y_0) $\frac{d(x_{2n},x_{2n}),d(x_{2n+1},x_{2n-1})+d(x_{2n},x_{2n-1})d(x_{2n+1},x_{2n})}{1+d(x_{2n-1},x_{2n})}$)

$\lesssim$ F (λ (x_0,y_0) d (x_{2n},x_{2n-1}) + μ (x_0,y_0) d (x_{2n+1},x_{2n})

+ δ (x_0, y_0)$\frac{d(x_{2n},x_{2n-1})d(x_{2n+1},x_{2n})}{1+d(x_{2n-1},x_{2n})}$) (1)

As F is strictly increasing, we deduce the following

|d (x_{2n}, x_{2n+1}) |≤ | (λ (x_0, y_0)) || d (x_{2n},x_{2n-1}) |+| μ (x_0, y_0)) ||d (x_{2n+1},x_{2n}) |

+| δ (x_0, y_0))| |$\frac{|d(x_{2n},x_{2n-1})d(x_{2n+1},x_{2n})|}{|1+d(2x_{n-1},x_{2n})|}$

∵ | 1 + $d(x_{2n-1}, x_{2n})$ |≥ | $d(x_{2n-1}, x_{2n})$ |

|d (x_{2n}, x_{2n+1}) |< | λ (x_0,y_0) || d (x_{2n},x_{2n-1}) |+ | μ (x_0, y_0)) ||d (x_{2n+1},x_{2n}) |

+| δ (x_0, y_0))| | d(x_{2n+1}, x_{2n}) |

|d (x_{2n}, x_{2n+1}) |< | (λ (x_0, y_0)) || d (x_{2n},x_{n-1}) |+|(μ (x_0, y_0)) + δ (x_0 , y_0)) ||d (x_{2n+1},x_{2n}) |

|d (x_{2n}, x_{2n+1}) | < $\frac{|\lambda(x_0,y_0)|}{|(1-\mu(x_0,y_0)-\delta(x_0,y_0)|}$ | d (x_{2n},x_{2n-1}) | = |d (x_{2n},x_{2n-1}) |

Consequently, from (1)

τ + F (d (x_{2n}, x_{2n+1}))< F (d (x_{2n},x_{2n-1}))

Thus

F (d (x_n, x_{n+1})) < F (d (x_{n-1},x_n)) - τ < F (d (x_{n-2},x_{n-1})) - 2τ < . . .

< F (d (x_0,x_1)) - n τ
(2)

For all n Є N. By taking limit as n → ∞ in (2) we obtain

$\lim_{n\to\infty}$ F (d (x_n, x_{n+1})) = - ∞ ⟺ $\lim_{n\to\infty}$ d (x_n, x_{n+1}) = 0 [by (F2)]

Now from (F3), there exists k Є (0, 1) such that $\lim_{n\to\infty}$ [d (x_n, x_{n+1})]^k F (d (x_n, x_{n+1})) = 0

By (2) the following holds for all n Є N

d $(x_n, x_{n+1})^k$ F [d (x_n, x_{n+1})) − d $(x_n, x_{n+1})^k$ F (d (x_1, x_0)]

< d $(x_n, x_{n+1})^k$ [F (d (x_0, x_1) − $n\tau$) - F (d(x_0, x_1)]

= $-n\tau$d $(x_n, x_{n+1})^k \le 0$ (3)

by taking limit as n → ∞ in (3) and applying (1) and (2) we obtain

$\lim_{n\to\infty}$ n [d (x_n, x_{n+1})]^k = 0 (4)

If follows from (4) that there exists n_1 Є ℕ such that

n [d (x_n, x_{n+1})]^k ≤ 1 for all n > n_1. This implies that

d (x_n, x_{n+1}) ≤ $\frac{1}{n^{1/k}}$ (5)

for all n > n_1. Now we prove that $\{x_n\}$ is a Cauchy sequence .For m > n > n_1, we have

d (x_m, x_n) ≤ $\sum_{i=n}^{m-1}$ d (x_i, x_{i+1}) ≤ $\sum_{i=n}^{m-1} \frac{1}{i^{1/k}}$ (6)

since k Є (0,1) ,then $\sum_{i=n}^{\infty} \frac{1}{i^{1/k}}$ converges. Therefore d(x_m, x_n) → 0 as m, n → ∞.thus $\{x_n\}$ is a Cauchy sequence in X. As X is a complete metric space there exists x Є X such that x_n → x as n → ∞.First we show that z is a Coupled fixed point of T.

τ + F(d (T(x,y),x)≲ τ + F(d(,T(x,y), x_{2n+1}) + d(x_{2n+1} , x)

= τ + F(d (T(x_{2n}, y_{2n}),T(x,y)) + d(x_{2n+1} , x)

 DOI : 10.35291/2454-9150.2019.0048

$= \tau + F($ d $T((x_{2n}, y_{2n}), T(x,y)) +$ d(x_{2n+1}, x)

$\lesssim F(\lambda(x_{2n},y_{2n})$ d$(T(x_{2n},y_{2n}),x_{2n}) + \mu(x_{2n},y_{2n})$ d$(T(x,y),x)$

$+ \delta(x_{2n},y_{2n})\frac{d(T(x_{2n},y_{2n}),x), d(T(x,y),x_{2n}) + d(T(x_{2n},y_{2n}),x_{2n}) d(T(x,y),x)}{1+d(x_{2n},x)}$) + d (x_{2n+1}, x)

$= F(\lambda(x_{2n},y_{2n})$ d $(x_{2n+1},x_{2n}) + \mu(x_{2n},y_{2n})$ d $(T(x,y),x)$

$+ \delta(x_{2n},y_{2n})\frac{d(x_{2n+1},x) d(T(x,y),x_{2n}), + d(x_{2n+1},x_{2n}) d(T(x,y),x)}{1+d(x_{2n},x)}$) + d (x_{2n+1}, x)

As F is strictly increasing, we deduce the following

d $(T(x,y),x) \lesssim \lambda(x_{2n},y_{2n})$ d $(x_{2n+1},x_{2n}) + \mu(x_{2n},y_{2n})$ d $(T(x,y),x)$

$+ \delta(x_{2n},y_{2n})\frac{d(x_{2n+1},x) d(T(x,y),x_{2n}), + d(x_{2n+1},x_{2n}) d(T(x,y),x)}{1+d(x_{2n},x)}$) + d (x_{2n+1}, x).

As $n \to \infty$. d $(T(x,y),x) = 0$.

Or $T(x,y) = x$.

Similarly we can show that $T(y,x) = y$.

Thus T has coupled fixed point.

Finally we will show that (x,y) is unique coupled fixed point of T.

If possible let (x^*, y^*) is another coupled fixed point of T then we have

$\tau + F(d(x, x^*) = \tau + F(d(T(x,y), T(x^*, y^*))$

$\tau + F(d(T(x,y), T(x^*, y^*)) \lesssim F(\lambda(x,y)\, d(T(x,y),x) + \mu(x,y)\, d(T(x^*, y^*), x^*)$

$+\delta(x,y)\frac{d(T(x,y),x^*), d(T(x^*,y^*),x) + d(T(x,y),x) d(T(x^*,y^*),x^*)}{1+d(x,x^*)}$)

$\lesssim F((x,y)\frac{d(T(x,y),x^*), d(T(x^*,y^*),x) + 0}{1+d(x,x^*)}$)

As F is strictly increasing, we deduce the following

d $(T(x,y), T(x^*, y^*)) \lesssim \delta(x,y)\frac{d(T(x,y),x^*), d(T(x^*,y^*),x)}{1+d(x,x^*)}$

$\lesssim \delta(x,y)\frac{d(x,x^*), d(x^*,x)}{1+d(x,x^*)}$

$(1 - \delta)(x,y)$ d$(x, x^*) \lesssim 0$.

d$(x, x^*) = 0$.

Thus we have $x = x^*$.similarly we can prove $y = y^*$.

Therefore T has unique coupled fixed point.

REFERENCES

[1] Abbas, M, Ali, B, Romaguera, S: Fixed and periodic points of generalized contractions in metric spaces. Fixed Point Theory Appl. 2013, 243 (2013)

[2] Azam, A, Fisher, B, Khan, M: Common fixed point theorems in complex valued metric spaces. Numer. Funct. Anal. Optim. 32(3), 243-253 (2011)

[3] Banach S., sur les operations dans les ensembles abs traits et leur application aux equations integrals, fund. Math.3 (1922), 133-181.1, 1.1

[4] Batra, R, Vashistha, S, Kumar, R: A coincidence point theorem for F-contractions on metric spaces equipped with an altered distance. J. Math. Comput. Sci. 4(5), 826-833 (2014)

[5] Batra, R, Vashistha, S: Fixed points of an F-contraction on metric spaces with a graph. Int. J. Comput. Math. 91, 1-8 (2014)

[6] Bhaskar T.G. and Lakshmikantham V.. Fixed point theorems in partially ordered metric spaces and applications. Nonlinear Analysis: Theory Methods is Applications, 65(7), 1379-1393, 2006

[7] Cosentino, M, Vetro, P: Fixed point results for F-contractive mappings of Hardy-Rogers-type. Filomat 28(4), 715-722 (2014)

[8] Hussain, N, Salimi, P: Suzuki-Wardowski type fixed point theorems for α-GF-contractions. Taiwan. J. Math. (2014).doi:10.11650/tjm.18.2014.4462

[9] Sgroi, M, Vetro, C: Multi-valued F-contractions and the solution of certain functional and integral equations. Filomat 27(7), 1259-1268 (2013)

[10] Wardowski, D, Van Dung, N: Fixed points of F-weak contractions on complete metric space. Demonstr. Math. XLVII(1),146-155 (2014)

[11] Wardowski, D: Fixed points of a new type of contractive mappings in complete metric spaces. Fixed Point Theory Appl. 2012, 94 (2012).

 DOI : 10.35291/2454-9150.2019.0048