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A Class Of 2-Step Fourth Order Rational Multi-Step Methods For Solving Delay Differential Equations

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Abstract: In this paper, a class of 2-step fourth order rational multi-step methods (RMMs) that are based on rational interpolating functions has been presented to solve delay differential equations (DDEs). The delay argument is approximated using Lagrange interpolation. The local truncation error and stability analysis of these methods are presented. The efficiency of these methods has been compared through three numerical examples of stiff and non-stiff DDEs with constant delay, time dependent delay and state dependent delays.

Keywords: Stiff and Non-Stiff Delay differential equations, Rational Multi-step methods, Lagrange interpolation, Stability polynomial and Stability region.

I. INTRODUCTION

Delay differential equations play a vital role in control systems [1], population dynamics [2], chemical kinetics [3] and in several areas of science and engineering. The nature of many models in differential equations are 'stiff'. Also, most of the problems are stiff in some intervals and non-stiff in other intervals. Solving non-stiff problems with stiff methods is very expensive, whereas non-stiff methods are more suitable for this purpose. Therefore, we need an efficient technique that are suitable for solving stiff and non-stiff problems.

In recent years there has been a great interest in finding the numerical solutions of DDEs. Some of the notable numerical methods are Homotopy perturbation method [4], Adomian decomposition method [5], Block method [6], combination of Laplace and variational iteration method [7], RKCeM method [8] and one step rational method [9].

There are many techniques used to solve stiff delay differential equations. Some of them are linear multi-step method [10], Chebyschev method [11], Parallel two-step ROW method [12] and one-step polynomial method [13].

Several multi-step techniques using different interpolating polynomials and functions have been developed to solve ODEs. Simeon [14] proposed nonlinear multi-step methods for IVPs based on inverse polynomial scheme. Okosun and Ademiluyi [15] developed two-step second order inverse polynomial methods for integration of differential equations with singularities. Teh Yuan Ying [16] derived a new class of rational multi-step methods for solving IVPs.

In this paper we present a class of 2-step fourth order multi-step methods based on rational interpolating functions discussed in [14], [15] and [16]. Throughout the paper we have denoted these rational multi-step methods as RMM1, RMM2 and RMM3. The local truncation error and stability analysis for each method are presented. The numerical examples are provided to compare the efficiency of these methods.

II. CONSTRUCTION OF RMM1

Consider the first order DDEs with delay τ,

$$y'(t) = f\big(t, y(t), y(t-\tau)\big), \quad t > t_0$$

$$y(t) = \Phi(t), \quad t \le t_0 \qquad (1)$$

where $\Phi(t)$ is the initial function and $\tau = \tau(t, y(t))$.

For 2-step fourth order RMM1, we assume an approximation to the analytical solution $y(t_{n+2})$ of (1) given by

$$y_{n+2} = \frac{A}{1+\sum_{j=1}^{4} b_j h^j}, \quad 1 + \sum_{j=1}^{4} b_j h^j \neq 0 \qquad (2)$$

where A, b_j, *(j=1,2,3,4)* are parameters that may contain approximations of $y(t_n)$ and higher derivatives of $y(t_n)$.

With RMM1 in (2) we associate the difference operator L defined by

$$L[y(t);h]_{RMM1} = \big(y(t+2h)\big) \times \big(1 + \sum_{j=1}^{4} b_j h^j\big) - A \qquad (3)$$

where y(t) is an arbitrary, continuous and differentiable function.

Expanding $y(t+2h)$ as Taylor series and collecting terms in (3), we have

$$L[y(t);h]_{RMM1} = C_0 h^0 + C_1 h^1 + \cdots + C_5 h^5 + \cdots \qquad (4)$$

where $C_i, \ i = 0,1,\ldots,5$ are the coefficients that need to be determined.

Expanding $y(t+2h)$ into Taylor series, (3) becomes

$$L[y(t);h]_{RMM1} = -A + y(t) + h\big(b_1 y(t) + 2y'(t)\big)$$
$$+h^2\big(2b_1 y'(t) + 2y''(t) + b_2 y(t)\big)$$

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$$+h^3\left(2b_1y''(t)+\frac{4}{3}y'''(t)+2b_2y'(t)+b_3y(t)\right)$$

$$+h^4\left(2b_2y''(t)+\frac{4}{3}b_1y'''(t)+\frac{2}{3}y^{(4)}(t)+2b_3y'(t)+b_4y(t)\right)$$

$$+h^5\left(\frac{4}{15}y^{(5)}(t)+2b_3y''(t)+2b_4y'(t)+\frac{4}{3}b_2y'''(t)+\frac{2}{3}b_1y^{(4)}(t)\right)$$

$$+o(h^6) \qquad (5)$$

Comparing (4) and (5), we have

$$C_0=-A+y(t),$$

$$C_1=b_1y(t)+2y'(t),$$

$$C_2=2b_1y'(t)+2y''(t)+b_2y(t),$$

$$C_3=2b_1y''(t)+\frac{4}{3}y'''(t)+2b_2y'(t)+b_3y(t),$$

$$C_4=2b_2y''(t)+\frac{4}{3}b_1y'''(t)+\frac{2}{3}y^{(4)}(t)+2b_3y'(t),$$

$$C_5=\frac{4}{15}y^{(5)}(t)-\frac{8y'(t)y^{(4)}(t)}{3y(t)}-\frac{16y''(t)y'''(t)}{3y(t)}+\frac{32\left(y'(t)\right)^5}{\left(y(t)\right)^4}+\frac{48\left(y'(t)\right)^2y'''(t)}{3\left(y(t)\right)^2}+\frac{24\left(y''(t)\right)^2y'(t)}{\left(y(t)\right)^2}-\frac{64\left(y'(t)\right)^3y''(t)}{\left(y(t)\right)^3} \qquad (6)$$

For fourth order RMM1, we put $C_0=C_1=\ldots=C_4=0$ in (6) and get the following solutions:

$$A=y(t),$$

$$b_1=\frac{-2y'(t)}{y(t)},$$

$$b_2=-\frac{2y''(t)}{y(t)}+\frac{4(y'(t))^2}{(y(t))^2},$$

$$b_3=\frac{8y'(t)y''(t)}{(y(t))^2}-\frac{4y'''(t)}{3y(t)}-\frac{8(y'(t))^3}{(y(t))^3},$$

$$b_4=\frac{16y'(t)y'''(t)}{3(y(t))^2}+\frac{4(y''(t))^2}{(y(t))^2}+\frac{16(y'(t))^4}{(y(t))^4}-\frac{2y^{(4)}(t)}{3y(t)}-\frac{24(y'(t))^2y''(t)}{(y(t))^3}$$

and $C_5=\frac{4}{15}y^{(5)}(t)-\frac{8y'(t)y^{(4)}(t)}{3y(t)}-\frac{16y''(t)y'''(t)}{3y(t)}$

$$+\frac{32\left(y'(t)\right)^5}{\left(y(t)\right)^4}+\frac{48\left(y'(t)\right)^2y'''(t)}{3\left(y(t)\right)^2}+\frac{24\left(y''(t)\right)^2y'(t)}{\left(y(t)\right)^2}-\frac{64\left(y'(t)\right)^3y''(t)}{\left(y(t)\right)^3} \qquad (7)$$

If we write $y_n=y(t_n)$ and $y_n^{(m)}=y^{(m)}(t_n)$ for m=1, 2,…, then (7) becomes

$$A=y_n,$$

$$b_1=\frac{-2y_n'}{y_n},$$

$$b_2=-\frac{2y_n''}{y_n}+\frac{4(y_n')^2}{y_n^2},$$

$$b_3=\frac{8y_n'y_n''}{y_n^2}-\frac{4y_n'''}{3y_n}-\frac{8(y_n')^3}{y_n^3},$$

$$b_4=\frac{16y_n'y_n'''}{3y_n^2}+\frac{4(y_n'')^2}{y_n^2}+\frac{16(y_n')^4}{y_n^4}-\frac{2y_n^{(4)}}{3y_n}-\frac{24(y_n')^2y_n''}{y_n^3} \qquad (8)$$

and

$$C_5=\frac{4}{15}y_n^{(5)}-\frac{8y_n'y_n^{(4)}}{3y_n}-\frac{16y_n''y_n'''}{3y_n}+\frac{32(y_n')^5}{(y_n)^4}+\frac{48(y_n')^2y_n'''}{3y_n^2}+\frac{24(y_n'')^2y_n'}{y_n^2}-\frac{64(y_n')^3y_n''}{y_n^3}$$

Substituting (8) in (2), we get

$$y_{n+2}=\frac{3y_n^5}{\begin{array}{c}3y_n^4-6hy_n^3y_n'+h^2(-6y_n''y_n^3+12y_n^2(y_n')^2)\\+h^3(24y_n'y_n''y_n^2-4y_n'''y_n^3-24(y_n')^3y_n)\\+h^4\begin{pmatrix}16y_n'y_n'''y_n^2+12(y_n'')^2y_n^2+48(y_n')^4\\-2y_n^{(4)}y_n^3-72(y_n')^2y_n''\end{pmatrix}\end{array}} \qquad (9)$$

The local truncation error (LTE) of RMM1 (2, 4) is given by

$$LTE_{RMM1(2,4)}=h^5\left(\frac{4}{15}y_n^{(5)}-\frac{8y_n'y_n^{(4)}}{3y_n}-\frac{16y_n''y_n'''}{3y_n}+\frac{32(y_n')^5}{(y_n)^4}+\frac{48(y_n')^2y_n'''}{3y_n^2}+\frac{24(y_n'')^2y_n'}{y_n^2}-\frac{64(y_n')^3y_n''}{y_n^3}\right)+o(h^6)$$

III. CONSTRUCTION OF RMM2

For 2-step fourth order RMM2, we assume an approximation to the analytical solution $y(t_{n+2})$ of (1) given by

$$y_{n+2}=a_0+\cfrac{a_1h}{1+\cfrac{a_2h}{1+\cfrac{a_3h}{1+\cdots\vdots\cfrac{a_kh}{1+a_5h}}}} \qquad (10)$$

where a_i for $i=0,1,\ldots,5$ are parameters that may contain approximations of $y(t_n)$ and higher derivatives of $y(t_n)$.

The simplified version of (10) is

$$y_{n+2}=a_0+\frac{P(a_j,h)}{Q(a_j,h)}, \qquad (11)$$

where $P(a_j,h)$ and $Q(a_j,h)$ are functions that contain the parameters a_j for $j=1,2,\ldots,5$.

With RMM2 in (11) we associate the difference operator L defined by

$$L[y(t);h]_{RMM2}=(y(t+2h)-a_0)\times Q(a_j,h)-P(a_j,h) \qquad (12)$$

where y(t) is an arbitrary, continuous and differentiable function.

Expanding $y(t+2h)$ as Taylor series and collecting terms in (12), we have

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$$L[y(t);h]_{RMM2} = C_0h^0 + C_1h^1 + \cdots + C_5h^5 + \cdots \qquad (13)$$

where C_i, $i = 0,1,\ldots,5$ are the coefficients that need to be determined.

Expanding $y(t+2h)$ to Taylor series, (12) becomes

$$\begin{aligned}L[y(t);h]_{RMM2} = {} & -a_0 + y(t) \\ & +h\big(-a_1 - a_0a_2 - a_0a_3 - a_0a_4 + a_2y(t) + a_3y(t) + a_4y(t) + 2y'(t)\big) \\ & + h^2\big(-a_1a_3 - a_1a_4 - a_0a_2a_4 + a_2a_4y(t) + 2a_2y'(t) + 2a_3y'(t) + 2a_4y'(t) + 2y''(t)\big) \\ & + h^3\left(2a_2a_4y'(t) + 2a_2y''(t) + 2a_3y''(t) + 2a_4y''(t) + \tfrac{4}{3}y'''(t)\right) \\ & +h^4\left(2a_2a_4y''(t) + \tfrac{4}{3}a_2y'''(t) + \tfrac{4}{3}a_3y'''(t) + \tfrac{4}{3}a_4y'''(t) + \tfrac{2}{3}y^{(4)}(t)\right) \\ & +h^5\left(\tfrac{4}{3}a_2a_4y'''(t) + \tfrac{2}{3}a_2y^{(4)}(t) + \tfrac{2}{3}a_3y^{(4)}(t) + \tfrac{2}{3}a_4y^{(4)}(t) + \tfrac{4}{15}y^{(5)}(t)\right) \\ & +o(h^6) \qquad (14)\end{aligned}$$

Comparing (13) and (14), we have

$$C_0 = -a_0 + y(t),$$

$$C_1 = -a_1 - a_0a_2 - a_0a_3 - a_0a_4 + a_2y(t) + a_3y(t) + a_4y(t) + 2y'(t),$$

$$C_2 = -a_1a_3 - a_1a_4 - a_0a_2a_4 + a_2a_4y(t) + 2a_2y'(t) + 2a_3y'(t) + 2a_4y'(t) + 2y''(t),$$

$$C_3 = 2a_2a_4y'(t) + 2a_2y''(t) + 2a_3y''(t) + 2a_4y''(t) + \tfrac{4}{3}y'''(t),$$

$$C_4 = 2a_2a_4y''(t) + \tfrac{4}{3}a_2y'''(t) + \tfrac{4}{3}a_3y'''(t) + \tfrac{4}{3}a_4y'''(t) + \tfrac{2}{3}y^{(4)}(t),$$

$$C_5 = \tfrac{4}{3}a_2a_4y'''(t) + \tfrac{2}{3}a_2y^{(4)}(t) + \tfrac{2}{3}a_3y^{(4)}(t) + \tfrac{2}{3}a_4y^{(4)}(t) + \tfrac{4}{15}y^{(5)}(t) \qquad (15)$$

For fourth order RMM2, we put $C_0 = C_1 = \ldots = C_4 = 0$ in (15), and get the following solutions.

$$a_0 = y(t),$$

$$a_1 = 2y'(t),$$

$$a_2 = -\frac{y''(t)}{y'(t)},$$

$$a_3 = \frac{3(y''(t))^2 - 2y'(t)y'''(t)}{3y'(t)y''(t)},$$

$$a_4 = \frac{-4y'(t)(y'''(t))^2y'''(t) + 3y'(t)y''(t)y^{(4)}(t)}{3(y''(t))^3 - 6y'(t)y''(t)y'''(t)} \qquad (16)$$

and

$$C_5 = \frac{16(y'''(t))^3 - 24y''(t)y'''(t)y^{(4)}(t) + 6y'(t)\left(y^{(4)}(t)\right)^2}{27(y''(t))^2 - 18y'(t)y'''(t)} + \frac{4}{15}y^{(5)}(t)$$

If we write $y_n = y(t_n)$ and $y_n^{(m)} = y^{(m)}(t_n)$ for m=1,2,…, then (16) becomes

$$a_0 = y_n,$$

$$a_1 = 2y_n',$$

$$a_2 = -\frac{y_n''}{y_n'},$$

$$a_3 = \frac{3(y_n'')^2 - 2y_n'y_n'''}{3y_n'y_n''}$$

$$a_4 = \frac{-4y_n'(y_n''')^2 + 3y_n'y_n''y_n^{(4)}}{3y_n''(3(y_n'')^2 - 2y_n'y_n''')}$$

And

$$c_5 = \frac{16\left(y_n'''\right)^3 - 24y_n''y_n'''y_n^{(4)}\left(y_n'\right)^2 + 6y_n'(y_n^{(4)})^2}{27\left(y_n''\right)^2 - 18y_n'y_n'''} + \frac{4}{15}y_n^{(5)} \qquad (17)$$

Substituting (17) in (11), we get

$$y_{n+2} = y_n + \frac{2h(y_n')^2}{y_n' - hy_n''} + \frac{2h^3(-3(y_n'')^2 + 2y_n'y_n''')^2}{(-y_n' + hy_n'')(9(y_n'')^2 - 6y_n'y_n'' - 6hy_n''y_n''' + 4h^2(y_n''')^2 + 3hy_n'y_n^{(4)} - 3h^2y_n''y_n''')} \qquad (18)$$

The local truncation error of RMM2 (2, 4) is given by,

$$LTE_{RMM2(2,4)} = h^5\left(\frac{16(y_n''')^3 - 24y_n''y_n'''y_n^{(4)}(y_n'')^2 + 6y_n'{y_n^{(4)}}^2}{27(y_n'')^2 - 18y_n'y_n'''} + \frac{4}{15}y_n^{(5)}\right) + o(h^6)$$

IV. CONSTRUCTION OF RMM3

For 2-step fourth order RMM3, we assume an approximation to the analytical solution $y(t_{n+2})$ of (1) given by

$$y_{n+2} = a_0 + \frac{a_1h}{1+\sum_{j=1}^{3} b_jh^j}, \quad 1 + \sum_{j=1}^{K} b_jh^j \neq 0 \qquad (19)$$

where a_0, a_1, b_j, *(j=1,2,3)* are parameters that may contain approximations of $y(t_n)$ and higher derivatives of $y(t_n)$.

With RMM3 in (19) we associate the difference operator L defined by

$$L[y(t);h]_{RMM3} = (y(t+2h) - a_0) \times \left(1 + \sum_{j=1}^{3} b_jh^j\right) - a_1h \qquad (20)$$

where y(t) is an arbitrary, continuous and differentiable function.

Expanding $y(t+2h)$ as Taylor series, (20) becomes

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$$L[y(t);h]_{RMM3} = C_0h^0 + C_1h^1 + \cdots + C_5h^5 + \cdots \quad (21)$$

where C_i, $i = 0,1,\ldots,5$ are the coefficients that need to be determined.

Expanding $y(t+2h)$ into Taylor series, (20) becomes

$$\begin{aligned} L[y(t);h]_{RMM3} &= -a_0 + y(t) \\ &+h\left(-a_1 - a_0b_1 + b_1y(t) + 2y'(t)\right) \\ &+h^2\left(-a_0b_2 + b_2y(t) + 2b_1y'(t) + 2y''(t)\right) \\ &+h^3\left(-a_0b_3 + b_3y(t) + 2b_2y'(t) + 2b_1y''(t) + \frac{4}{3}y'''(t)\right) \\ &+h^4\left(2b_3y'(t) + 2b_2y''(t) + \frac{4}{3}b_1y'''(t) + \frac{2}{3}y^{(4)}(t)\right) \\ &+h^5\left(2b_3y''(t) + \frac{4}{3}b_2y'''(t) + \frac{2}{3}b_1y^{(4)}(t) + \frac{4}{15}y^{(5)}(t)\right) \\ &+o(h^6) \end{aligned} \quad (22)$$

Comparing (21) and (22), we get

$$\begin{aligned} C_0 &= -a_0 + y(t), \\ C_1 &= -a_1 - a_0b_1 + b_1y(t) + 2y'(t), \\ C_2 &= -a_0b_2 + b_2y(t) + 2b_1y'(t) + 2y''(t), \\ C_3 &= -a_0b_3 + b_3y(t) + 2b_2y'(t) + 2b_1y''(t) + \frac{4}{3}y'''(t), \\ C_4 &= 2b_3y'(t) + 2b_2y''(t) + \frac{4}{3}b_1y'''(t) + \frac{2}{3}y^{(4)}(t), \\ C_5 &= 2b_3y''(t) + +\frac{4}{3}b_2y'''(t) + \frac{2}{3}b_1y^{(4)}(t) + \frac{4}{15}y^{(5)}(t) \end{aligned} \quad (23)$$

For fourth order RMM3, we put $C_0 = C_1 = \ldots = C_4 = 0$ in (23) and get the following solutions:

$$\begin{aligned} a_0 &= y(t), \\ a_1 &= 2y'(t), \\ b_1 &= -\frac{y'''(t)}{y'(t)}, \\ b_2 &= \frac{3(y''(t))^2 - 2y'(t)y'''(t)}{3(y'(t))^2}, \\ b_3 &= \frac{-3(y''(t))^3 + 4y'(t)y''(t)y'''(t) - (y'(t))^2y^{(4)}(t)}{3(y'(t))^3} \end{aligned}$$

and

$$C_5 = \frac{2((9y''(t))^4 - 18y'(t)(y''(t))^2y'''(t) + 4(y'(t))^2(y'''(t))^2 + 6(y'(t))^2y''(t)y^{(4)}(t))}{9(y'(t))^3} + \frac{4}{15}y^{(5)}(t) \quad (24)$$

If we write $y_n = y(t_n)$ and $y_n^{(m)} = y^{(m)}(t_n)$, m=1,2,..,then (24) becomes

$$\begin{aligned} a_0 &= y_n, \\ a_1 &= 2y_n', \\ b_1 &= -\frac{y_n'''}{y_n'}, \\ b_2 &= \frac{3(y_n''')^2 - 2y_n'y_n'''}{3(y_n')^2}, \\ b_3 &= \frac{-3(y_n'')^3 + 4y_n'y_n''y_n''' - (y_n'')^2y_n^{(4)}}{3(y_n')^3} \end{aligned} \quad (25)$$

and

$$C_5 = \frac{2\left(9\left(y_n''\right)^4\right) - 18y_n'\left(y_n''\right)^2y_n''' + 4\left(y_n'\right)^2\left(y_n'''\right)^2 + 6\left(y_n'\right)^2y_n''y_n^{(4)}}{9(y_n')^3} + \frac{4}{15}y_n^{(5)}$$

Substituting (25) in (19), we get

$$y_{n+2} = y_n + \frac{6h(y_n')^4}{3(y_n')^3 - 3h(y_n')^2y_n'' + 3h^2y_n'(y_n'')^2 - 3h^3(y_n'')^2 - 2h^2(y_n')^2y_n''' + 4h^3y_n'y_n''y_n''' - h^3(y_n')^2y_n^{(4)}} \quad (26)$$

The local truncation error of RMM3 (2, 4) is given by,

$$LTE_{RMM3(2,4)} = h^5\left(\frac{2\left(9\left(y_n''\right)^4\right) - 18y_n'\left(y_n''\right)^2y_n''' + 4\left(y_n'\right)^2\left(y_n'''\right)^2 + 6\left(y_n'\right)^2y_n''y_n^{(4)}}{9(y_n')^3} + \frac{4}{15}y_n^{(5)}\right) + o(h^6)$$

V. STABILITY ANALYSIS OF THE RATIONAL MULTI-STEP METHODS

Consider a commonly used linear test equation with a constant delay $\tau = mh$, where m is a positive integer,

$$\begin{aligned} y'(t) &= \lambda y(t) + \mu y(t-\tau), \quad t > t_0 \\ y(t) &= \phi(t), \quad t \le t_0 \end{aligned} \quad (27)$$

where $\lambda, \mu \in C$, $\tau > 0$ and Φ is continuous.

A slight rearrangement of (9), (18) and (26) that corresponds to RMM1, RMM2 and RMM3 can be written as

$$y_{n+2} = y_n + 2hy_n' + 2h^2y_n'' + \frac{4}{3}h^3y_n''' + \frac{2}{3}h^4y_n^{(4)} \quad (28)$$

Hence we will have same stability polynomial and stability regions in these methods.

Using (28) in (27), we get

$$\begin{aligned} y_{n+2} = y_n &+ h\left(\lambda y_n + \mu y(t_n - \tau)\right) \\ &+ \frac{h^2}{2}\left(\lambda y_n' + \mu y'(t_n - \tau)\right) \\ &+ \frac{4}{3}h^3\left(\lambda y_n'' + \mu y''(t_n - \tau)\right) \\ &+ \frac{2}{3}h^4\left(\lambda y_n''' + \mu y'''(t_n - \tau)\right) \end{aligned} \quad (29)$$

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Taking $y(t_n-\tau)=\sum_{l=-r_1}^{s_1} L_l(c)y_{n-m+l}$,

$$y'(t_n-\tau)=\lambda\sum_{l=-r_1}^{s_1} L_l(c)y_{n-m+l} +\mu\sum_{l=-r_1}^{s_1} L_l(c)y_{n-2m+l},$$

$$y''(t_n-\tau)=\lambda\begin{pmatrix}\lambda\sum_{l=-r_1}^{s_1} L_l(c)y_{n-m+l}\\ +\mu\sum_{l=-r_1}^{s_1} L_l(c)y_{n-2m+l}\end{pmatrix} +\mu\begin{pmatrix}\lambda\sum_{l=-r_1}^{s_1} L_l(c)y_{n-2m+l}\\ +\mu\sum_{l=-r_1}^{s_1} L_l(c)y_{n-3m+l}\end{pmatrix}$$

and

$$y'''(t_n-\tau)=\lambda\begin{pmatrix}\lambda\begin{pmatrix}\lambda\sum_{l=-r_1}^{s_1} L_l(c)y_{n-m+l}\\ +\mu\sum_{l=-r_1}^{s_1} L_l(c)y_{n-2m+l}\end{pmatrix}\\ +\mu\begin{pmatrix}\lambda\sum_{l=-r_1}^{s_1} L_l(c)y_{n-2m+l}\\ +\mu\sum_{l=-r_1}^{s_1} L_l(c)y_{n-3m+l}\end{pmatrix}\end{pmatrix} +\mu\begin{pmatrix}\lambda\begin{pmatrix}\lambda\sum_{l=-r_1}^{s_1} L_l(c)y_{n-2m+l}\\ +\mu\sum_{l=-r_1}^{s_1} L_l(c)y_{n-3m+l}\end{pmatrix}\\ +\mu\begin{pmatrix}\lambda\sum_{l=-r_1}^{s_1} L_l(c)y_{n-3m+l}\\ +\mu\sum_{l=-r_1}^{s_1} L_l(c)y_{n-4m+l}\end{pmatrix}\end{pmatrix} \quad (30)$$

Then (30) becomes

$$y_{n+2}=y_n+2h\left(\lambda y_n+\mu\sum_{l=-r_1}^{s_1} L_l(c)y_{n-m+l}\right) +2h^2\begin{pmatrix}\lambda^2 y_n+2\lambda\mu\sum_{l=-r_1}^{s_1} L_l(c)y_{n-m+l}\\ +\mu^2\sum_{l=-r_1}^{s_1} L_l(c)y_{n-2m+l}\end{pmatrix} +\frac{4}{3}h^3\begin{pmatrix}\lambda^3 y_n+3\lambda^2\mu\sum_{l=-r_1}^{s_1} L_l(c)y_{n-m+l}\\ +3\lambda\mu^2\sum_{l=-r_1}^{s_1} L_l(c)y_{n-2m+l}\\ +\mu^3\sum_{l=-r_1}^{s_1} L_l(c)y_{n-3m+l}\end{pmatrix} +\frac{2}{3}h^4\begin{pmatrix}\lambda^3 y_n+4\lambda^3\mu\sum_{l=-r_1}^{s_1} L_l(c)y_{n-m+l}\\ +6\lambda^2\mu^2\sum_{l=-r_1}^{s_1} L_l(c)y_{n-2m+l}\\ +4\lambda\mu^3\sum_{l=-r_1}^{s_1} L_l(c)y_{n-3m+l}\\ +\mu^4\sum_{l=-r_1}^{s_1} L_l(c)y_{n-4m+l}\end{pmatrix}$$

$$y_{n+2}=y_n+2\lambda h y_n+2\lambda^2h^2y_n+\frac{4}{3}h^3y_n+\frac{2}{3}h^4y_n +\sum_{l=-r_1}^{s_1} L_l(c)y_{n-m+l}\left(2\mu h+2h^2\mu\lambda+4h^3\lambda^2\mu+\frac{8}{3}h^4\lambda^3\mu\right) +\sum_{l=-r_1}^{s_1} L_l(c)y_{n-2m+l}\left(2h^2\mu^2+4h^3\lambda\mu^2+4h^4\lambda^2\mu^2\right) +\sum_{l=-r_1}^{s_1} L_l(c)y_{n-3m+l}\left(\frac{4}{3}h^3\mu^3+\frac{8}{3}h^4\lambda\mu^3\right) +\sum_{l=-r_1}^{s_1} L_l(c)y_{n-4m+l}\left(\frac{2}{3}h^4\mu^4\right)$$

$$y_{n+2}=y_n\left(1+2\lambda h+2(\lambda h)^2+\frac{4}{3}(\lambda h)^3+\frac{2}{3}(\lambda h)^4\right) +\sum_{l=-r_1}^{s_1} L_l(c)y_{n-m+l}\left(\mu h\left(2+2\lambda h+4(\lambda h)^2+\frac{8}{3}(\lambda h)^3\right)\right) +\sum_{l=-r_1}^{s_1} L_l(c)y_{n-2m+l}\left((\mu h)^2(2+4\lambda h+4(\lambda h)^2)\right) +\sum_{l=-r_1}^{s_1} L_l(c)y_{n-3m+l}\left((\mu h)^3\left(\frac{4}{3}+\frac{8}{3}\lambda h\right)\right) +\sum_{l=-r_1}^{s_1} L_l(c)y_{n-4m+l}\left(\frac{2}{3}(\mu h)^4\right)$$

Let $\alpha=\lambda h$ and $\beta=\mu h$ then the above equation becomes

$$y_{n+2}=y_n\left(1+2\alpha+2\alpha^2+\frac{4}{3}\alpha^3+\frac{2}{3}\alpha^4\right) +\sum_{l=-r_1}^{s_1} L_l(c)y_{n-m+l}\left(\beta\left(2+4\alpha+4\alpha^2+\frac{8}{3}\alpha^3\right)\right) +\sum_{l=-r_1}^{s_1} L_l(c)y_{n-2m+l}\left(\beta^2(2+4\alpha+4\alpha^2)\right) +\sum_{l=-r_1}^{s_1} L_l(c)y_{n-3m+l}\left(\beta^3\left(\frac{4}{3}+\frac{8}{3}\alpha\right)\right) +\sum_{l=-r_1}^{s_1} L_l(c)y_{n-4m+l}\left(\frac{2}{3}\beta^4\right)$$

To obtain the stability polynomial, the delay term is approximated using five points Lagrange interpolation. By putting

$n-m+l=0$, $n-2m+l=0$, $n-3m+l=0$ and $n-4m+l=0$ and by taking l = -1, 0, 1, 2, 3 the stability polynomial will be in the standard form. When $\tau=1$, the recurrence is stable if the zeros of ζ_i of the stability polynomial

$$S(\alpha,\beta:\zeta)=\zeta^{n+2}-\left(1+\alpha+2\alpha^2+\frac{4}{3}\alpha^3+\frac{2}{3}\alpha^4\right)\zeta^n -\beta\left(2+4\alpha+4\alpha^2+\frac{8}{3}\alpha^3\right)(L_{-1}(c)+L_0(c)\zeta+L_1(c)\zeta^2+L_2(c)\zeta^3+L_3(c)\zeta^4) -\beta^2(2+4\alpha+4\alpha^2)(L_{-1}(c)+L_0(c)\zeta+L_1(c)\zeta^2+L_2(c)\zeta^3+L_3(c)\zeta^4) -\beta^3\left(\frac{4}{3}+\frac{8}{3}\alpha\right)(L_{-1}(c)+L_0(c)\zeta+L_1(c)\zeta^2+L_2(c)\zeta^3+L_3(c)\zeta^4) -\left(\frac{2}{3}\beta^4\right)(L_{-1}(c)+L_0(c)\zeta+L_1(c)\zeta^2+L_2(c)\zeta^3+L_3(c)\zeta^4)$$

satisfies the root condition $|\zeta_i|\le 1$.

Then the stability polynomial for the method is given as

$$S(\alpha,\beta:\zeta)=\zeta^{n+2} -\left(1+\alpha+2\alpha^2+\frac{4}{3}\alpha^3+\frac{2}{3}\alpha^4\right)\zeta^n -\left(2\beta+2\beta^2+\frac{4}{3}\beta^3+\frac{2}{3}\beta^4+2\alpha\beta+4\alpha^2\beta+4\alpha\beta^2+\frac{8}{3}\alpha\beta^3+\frac{8}{3}\alpha^3\beta+4\alpha^2\beta^2\right)$$

The corresponding stability region is given in Fig. 1.

VI. NUMERICAL EXAMPLES

Problem 1: (Stiff linear system with multiple delays)

$$y_1'(t)=-\frac{1}{2}y_1(t)-\frac{1}{2}y_2(t-1)+f_1(t),$$

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$$y_2'(t) = -y_2(t) - \frac{1}{2}y_1\left(t-\frac{1}{2}\right) + f_2(t), 0 \le t \le 1$$

with initial conditions

$$y_1(t) = e^{-t/2}, \qquad \frac{-1}{2} \le t \le 0,$$

$$y_2(t) = e^{-t}, \qquad -1 \le t \le 0$$

and $f_1(t) = \frac{1}{2}e^{-(t-1)}$, $f_2(t) = \frac{1}{2}e^{-(t-1/2)/2}$

The exact solution is

$$y_1(t) = e^{-t/2}, \qquad y_2(t) = e^{-t}$$

.

Problem 2: (Time-dependent delay)

$$y'(t) = \frac{t-1}{t}y(\ln(t)-1)y(t), \quad 1 \le t \le \frac{3}{2}$$

with initial condition

$$y(t) = 1, \qquad 0 \le t \le 1$$

and the exact solution is

$$y(t) = \exp(t - \ln(t) - 1), \quad 1 \le t \le \frac{3}{2}$$

Problem 3: (State-dependent delay)

$$y'(t) = \cos(t)y(y(t)-2), \quad t \ge 0$$

with initial condition

$$y(t) = 1, \qquad t \le 0$$

and the exact solution is

$$y(t) = \sin(t) + 1, \qquad 0 \le t \le 1$$

By taking h = 0.01 in the above examples, absolute errors of RMM1, RMM2 and RMM3 are given in Tables 1 – 4.

VII. CONCLUSION

In this paper, a class of 2-step fourth order multi-step methods that are based on rational functions has been presented for solving delay differential equations. These methods have been referred here as RMM1, RMM2, and RMM3. In these methods, three types of rational functions have been considered for the approximation to the analytical solution. The local truncation errors of these methods have been determined. The stability polynomials of these methods are derived and obtained the stability region.

Numerical examples of stiff and non-stiff DDEs with constant delay, time dependent delay and state dependent delays have been considered to demonstrate the efficiency of these methods. The numerical results reveal that these multi-step methods are suitable to solve DDEs. While comparing the results, it is evident that among these three methods RMM3 gives more accurate results than the other two methods. These new class of rational multi-step methods are computationally efficient, robust and easy to implement.

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Table 1 Numerical results of y_1 in Example 1

t	Absolute error in RMM1(2,4)	Absolute error in RMM2(2,4)	Absolute error in RMM3(2,4)
0.2	7.477241e-12	9.057603e-07	1.256772e-12
0.4	1.353173e-11	1.639131e-06	2.274514e-12
0.6	1.836631e-11	2.224719e-06	3.086975e-12
0.8	2.215772e-11	2.684011e-06	3.724354e-12
1.0	2.506262e-11	3.035740e-06	4.212297e-12

Table 2 Numerical results of y_2 in Example 1

t	Absolute error in RMM1(2,4)	Absolute error in RMM2(2,4)	Absolute error in RMM3(2,4)
0.2	3.258821e-04	3.324783e-04	3.258823e-04
0.4	5.616800e-04	5.724818e-04	5.616803e-04
0.6	7.266744e-04	7.399408e-04	7.266748e-04
0.8	8.363701e-04	8.508532e-04	8.363705e-04
1.0	9.032073e-04	9.180306e-04	9.032077e-04

Table 3 Numerical results of Example 2

t	Absolute error in RMM1(2,4)	Absolute error in RMM2(2,4)	Absolute error in RMM3(2,4)
1.1	1.450523e-06	9.578036e-06	3.780994e-07
1.2	2.364011e-06	8.026518e-06	3.903088e-08
1.3	2.344030e-06	8.412107e-06	9.991727e-07
1.4	7.831233e-07	1.191906e-05	2.729523e-06
1.5	5.960803e-06	6.830788e-06	3.626918e-06

Table 4 Numerical results of Example 3

t	Absolute error in RMM1(2,4)	Absolute error in RMM2(2,4)	Absolute error in RMM3(2,4)
0.2	1.087553e-08	4.874883e-07	6.554333e-09
0.4	1.783060e-08	5.461268e-06	8.342122e-09
0.6	2.660190e-08	3.580044e-05	8.551573e-09
0.8	2.957885e-08	2.319097e-04	1.155962e-08
1.0	1.841471e-07	8.937141e-04	3.236336e-08

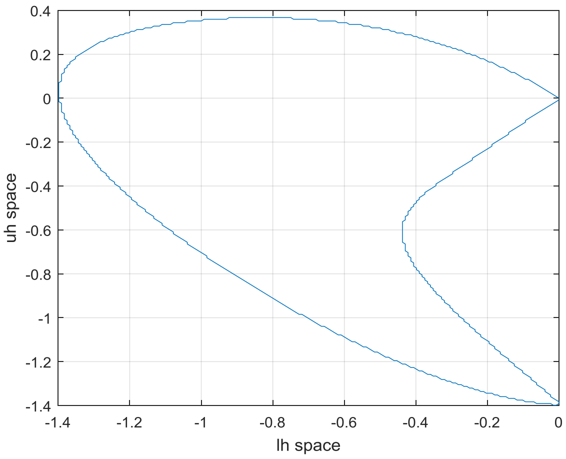

Fig. 1. Stability region for 2-step fourth order RMMs.

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