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Study of Spectrum of Second Order Differential Equations and Their Properties

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ABSTRACT - This paper presents about spectrum and its properties. The theory of eigen function expansions associated with the second order differential equations goes far back to the time of Sture and Liouville, i.e., more than a century ago. The modern theory of singular differential operator was first developed by N.Neyl (1885-1955) on singular self-and joint Linear differential operator of the second order and later on developed by M.H. Stone, J.Von Newmann (1905-1957), K. Friedrichs, K.Kodaira. Work on boundary value problems associated with self-adjoint differential system due to David Hilbert (1862-1941) was fundamental one. But the discussion on the simultaneous system was started by either in the early 20th century. Schlesinger took a system of a linear differential equations of the first order with coefficient to which one rational in X and obtained the asymptotic forms for a solution. Harwitz considered the simultaneous expansion of two functions in terms of solutions of a pair of differential equations of the first order with restricted boundary conditions. Mirkhoff and Langer and Bliss considered the possibilities of simultaneously expanding n arbitrary functions in terms of the solutions of a property restricted type of first order differential equations with a number of boundary conditions at one or both ends of a finite interval. Titchmarsh in 1944 discussed the finite case of the simultaneous system of two first-order linear differential equations and he considered the extension to the infinite case in 1941. Context Sangren discussed two first-order equations in 1953 and 1954. Since then Roos and sangren have worked as the same problem in a series of papers. Their methods are those of Ticharah's complex variable methods.

Keywords – Spectrum, Eigen function, Differential Equations.

I. INTRODUCTION

Titchmarah [52, chap V] has discussed the spectral theorems for a second order differential equation

$$\frac{d^2y}{dx^2} + \big(\lambda - q(x)\big)y = 0, \;\; 0 \leq x < \infty \tag{1}$$

Roos and Sangern in [47] have considered a pair of first order differential equations

$$x_1'(t) - \big(\lambda + q_1(t)\big)x_2(t) = 0$$
$$x_2'(t) - \big(\lambda + q_2(t)\big)x_1(t) = 0 \tag{2}$$

and proved some spectral theorems.

Bhagat in [5] has discussed the nature of the spectrum for the system

$$u''(x) + p(x)\,u(x) + r(x)\,v(x) = -\lambda u(x)$$
$$v''(x) + q(x)\,v(x) + r(x)\,u(x) = -\lambda v(x) \tag{3}$$

and prove that if p(x), q(x) and r(x) all belong to L [0,∞] then the spectrum for the system (3) in continuous in (0,∞) and there is a point spectrum in (-∞, 0) which is bounded below. Paladhi in has discussed the nature of the spectrum under different condition.

In this paper was shall discuss the nature of the spectrum of the system (1) under conditions different from those of their's. The methods followed are the same as that of Bhagat and Titchmarah. We also use the results and notations of Bhagat. Following Bhagat, the spectrum is defined as the complement of the set of points in the neighbourhood of which the matrix K(λ) is constant where K(λ) is defined in (1). Any point of discontinuity of K(λ) clearly belongs to the spectrum. The set of such points is the point spectrum. The continous spectrum is the set of the points where the matrix K (λ) is continuous but in the

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neighbourhood of which the matrix K(λ) is not constant, together with the derived set of this set. If this set is denoted by P, then P is closed and it may include the points of the point-spectrum.

II. TRANSFORMATION OF THE BASIC EQUATION

If can be verified that under the transformation

$$w(x) = \int_0^x (\lambda - p(t))^{½} dt\,,\ s(x) = (\lambda - p(t))^{\frac{1}{4}} u(x)$$

$$z(x) = \int_0^x (\lambda - q(t))^{½} dt\,,\ T(x) = (\lambda - p(t))^{\frac{1}{4}} u(x) \tag{4}$$

the differential system (2.1.1) reduces to

$$\frac{d^2 s}{dw^2} + S + P_1(x)S + R_1(x)T = 0$$

$$\frac{d^2 T}{dz^2} + T + Q_1(x)T + R_2(x)S = 0 \tag{5}$$

where

$$P_1(x) = \frac{5}{16}\frac{p'(x)^2}{(\lambda - p(x))^3} + \frac{1}{4}\frac{p"(x)}{(\lambda - p(x))^2}$$

$$Q_1(x) = \frac{5}{16}\frac{q'(x)^2}{(\lambda - q(x))^3} + \frac{1}{4}\frac{q"(x)}{(\lambda - q(x))^2}$$

$$R_1(x) = \frac{r(x)}{(\lambda - p(x))^{\frac{3}{4}}(\lambda - q(x))^{1/4}}$$

$$R_2(x) = \frac{r(x)}{(\lambda - p(x))^{\frac{1}{4}}(\lambda - q(x))^{3/4}}$$

$P_1(x)$, $Q_1(x)$, $R_1(x)$ and $R_2(x)$ and small when |λ| is large or when p(x) and q(x) are large. Now the integral equations for the system (2.2.2) are

$$S(w) = S(0)oos\, w + s'(0)\sin w - \int_0^w \{P_1(t)S(\tau) + R_1(t)T(\tau)\}\sin(w-\tau)\, d\tau$$

$$T(z) = T(0)oos\, z + T'(0)\sin z - \int_0^w \{Q_1(t)T(\tau) + R_2(t)S(\tau)\}\sin(z-\tau)\, d\tau \tag{6}$$

where τ =w(t) and τ_1= z(t)

It λ is not real, or p(x), q(x) > |λ|, w and z are not real, then (2.2.3) would involve integrals along complex path. To avoid this we proceed as follows.

There is no loss of generality if we take p(0) = q(0) - 0.

Let us take

$$v(x) = (\lambda - p(x))^{\frac{1}{4}}\frac{d}{dx}\left\{\left(\lambda - p(x)\right)^{-½}\frac{ds}{dx}\right\} - \frac{d^2u(x)}{dx^2} - \gamma(x)v(x)$$

$$= -\left[\frac{1}{4}\frac{p"(x)}{(\lambda - p(x))^2} + \frac{5}{16}\frac{p'(x)^2}{(\lambda - p(x))^2}\right]u(x) - \gamma(x)v(x)$$

Following Titchmarah [52, 5.4], we have

$$S_j(x,\lambda) = S_j(0)\cos w(x) + \frac{1}{\sqrt{\pi}}S_j'(0)\sin w(x) -$$

$$- \int_0^x \{p(t)S_j(t) + R(t)T_j(t)\}\sin(\mathrm{w(x) - w(t)})\,\mathrm{dt},$$

$$T_j(x,\lambda) = T_j\,(0)\cos z(x) + \frac{1}{\sqrt{\pi}}T_j'(0)\sin z(x) -$$

$$- \int_0^x \{Q(t)T_j(t) + R(t)S_j(t)\}\sin(\mathrm{z(x) - z(t)})\,\mathrm{dt},$$

$$(j = 1,2) \tag{7}$$

where

$$P(x) = \frac{1}{4}\frac{p"(x)}{(\lambda - p(x))^{3/2}}\ \frac{5}{16}\frac{p\prime(x)^2}{(\lambda - p(x))^{5/2}}$$

$$Q(x) = \frac{1}{4}\frac{q"(x)}{(\lambda - q(x))^{3/2}}\ \frac{5}{16}\frac{q\prime(x)^2}{(\lambda - q(x))^{5/2}}$$

$$\text{and}\ \ R(x) = \frac{r(x)}{(\lambda - p(x))^{1/4}\lambda - p(x))^{1/4}} \tag{8}$$

III. INTEGRAL SOLUTION

Let p(x) and q(x) tend to ∞ as x → ∞ such that p'(x) ≥ 0, q'(x) ≥ 0,

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p'(x) = O [$(p(x))^c$], $0 < c < \frac{3}{2}$

p'(x) = O [$(q(x))^{c1}$], $0 < c_1 < \frac{3}{2}$

and r(x) is bounded, or

r(x) = O [$(pq)^d$], $O < a < \frac{1}{4}$

and let p'' (x) be ultimately of one sign and so be q''(x).

Exactly following Titchmarah it can be proved that

$$\int_x^x \frac{p'(x)^2}{(p(x))^{5/2}} dx = O\ (1).$$
$$\int_x^x \frac{p''(x)}{(p(x))^{3/2}} dx = O\ (1).$$
$$\int_x^x \frac{q'(x)^2}{(p(x))^{5/2}} dx = O\ (1).$$
$$\int_x^x \frac{q''(x)}{(q(x))^{3/2}} dx = O\ (1).$$

Also

$$\int_{x_0}^x \frac{r(x)dx}{(p(x)q(x))^{1/4}} = O\left[\int_{x_0}^x \frac{dx}{(p(x)q(x))^{\frac{1}{4}} - d}\right] = 0(1)$$

are $x \to \infty$.

Thus $\int_0^\infty |p(x)|dx \quad \int_0^\infty |Q(x)|dx \quad and \quad \int_0^\infty |R(x)|dx$

are uniformly convergent when respect to λ over any region for which, $|\lambda - p(x)| \geq \delta > 0$, $|\lambda - q(x)| \geq \delta > 0$. for $0 \leq x < \infty$.

IV. SPECTRAL THEOREM

In this section we prove some of the spectral theorems for the system (1) under different conditions on the coefficients. These theorems are analogous to such theorems proved by Titchmarah for the second order differential equation.

Theorem

If p(x), q(x), r(x) satisfy the conditions of 3 and [w(x) ~ z(x)] = 0 (1) as $x \to \infty$ then the spectrum for the system (1) s discrete.

Let $\lambda = \mu + i\upsilon$ and $\upsilon > 0$, then $0 < \arg \lambda < \pi$

and $\arg \lambda \leq \arg (\lambda - p(x)) < \pi$

$\arg \ \leq \arg (\lambda - q(x)) < \pi$ (9)

If follow that

$\frac{1}{2} \arg \lambda \leq \arg (\lambda - p(x))^{\frac{1}{2}} < \pi/2$

and

$\frac{1}{2} \arg \lambda \leq \arg (\lambda - pq(x)) < \pi/2$

Let λ be bounded, as $x \to \infty$. We have

$w(x) \sim i \int_0^x (p(t))^{\frac{1}{2}} dt,\ z(x) \sim i \int_0^x (q(t))^{\frac{1}{2}} dt$ (10)

If follows from (10) that, as $x \to \infty$,

$e^{-iw(x)} \to \infty$, (11)

and

$e^{-z(x)} \to \infty$. (12)

Let $S_{ji}(x) = e^{iw(x)} S_j(x)$ and $T_{ji}(x) = e^{iz(x)} T_j(x)$, (j= 1,2). Then (12) becomes

$S_{j1}(x) = S_j(0)e^{iw(x)} cosw(x) + \frac{S_j'(0)}{\sqrt{\pi}} e^{iw(x)} sinw(x) - \int_0^x e^{i(w(x)-w(t))} \{p(t)S_{j1}(t) + R(t)T_{j1}(t)e^{i(w(t)-z(t))}\} \sin(w(x) - w(t)dt,$

$T_{j1}(x) = T_j(0)e^{iz(x)} cosz(x) + \frac{T_j'(0)}{\sqrt{\pi}} e^{iz(x)} sinz(x) - \int_0^x e^{i(z(x)-z(t))} \{Q(t)T_{j1}(t) + R(t)S_{j1}(t)e^{i(z(t)-z(t))}\} \sin(z(x) - z(t)dt,$

(j=1,2)

Now

$|\sin w(x)|$, $|\cos w(x)|$ < $e^{imw(x)}$

$|\sin z(x)|$, $|\cos z(x)|$< $e^{imz(x)}$ (13)

Let

$H(x) = \text{Max} \{ |p(x)|, |Q(x)|, |R(x)\ e^{i(w(x)-z(x))}|, |\mu(x)e^{i(z(x)-w(z))}| \}$

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and

$A = Max\{S_j(0), S_j'(0), T_j(0), T_j'(0)$

Therefore mains the condition [w(t) ~ z(t)] = O (1) and the inequalities of (13), we have for large x.

$$|S_{j1}(x)|, |T_{j1}(x)| \le A\left(1+\frac{1}{\sqrt{\lambda}}\right) + \int_0^x \{|S_{j1}(t)| + |T_{j1}(t)|\}H(t)dt \quad (14)$$

(j = 1,2)

We now require centre and Sangern's Lemma [14, p. 700] which states as lows:-

Let h_1, h_2 be integrable on $a \le x \le b$. Let h_1, h_2, g_1, g_2 be position with g_1, g_2 continuous and C be a constant. If $g_1(x), g_2(x) \le C + (g_1 + g_2h_2)$ dy, $a < x < b$,

then,

$g_1(x), g_2(x) \le C\{\int_a^x (h_1 + h_2)dy\}$

Using Lemma, we get from (14)

$$|S_{j1}(x)|, |T_{j1}(x)| \le A\left(1+\frac{1}{\sqrt{\lambda}}\right) exp\left\{2\int_0^x H(t)dt\right\} \le A\left(1+\frac{1}{\sqrt{\pi}}\right) exp\left\{2\int_0^\infty H(t)dt\right\}$$

So $S_{j1}(x)$ and $T_{j1}(x)$ are bounded for all x, in $w(x) \ge 0$, in $z(x) \ge 0$.

Thus for large x

$S_j(x) = O(e^{-iw(x)})$, (j = 1,2) (15)

and

$T_j(x) = O(e^{-iz(x)})$, (j = 1,2) (16)

uniformly with respect to λ.

From (16), we have

$$S_j(x) = \frac{1}{2}S_j(0)e^{-iw(x)} - \frac{1}{2i\sqrt{\lambda}}S_j'(0)e^{-iw(x)} + O\left(e^{iw(x)}\right) + \frac{1}{2i}\int_0^x e^{-i(w(x)-w(t))}\{p(t)S_j(t) + R(t)T_j(t)\}dt$$
$$+ O\{\int_0^x e^{i(w(x)-w(t))}\{p(t)S_j(t)R(t)T_j(t)\}dt\}$$

(j= 1, 2) (17)

as $x \to \infty$.

Now the last term in (17)

$= O\left[\int_0^y + \int_y^x e\right]$

$$= \left[\int_0^y |e^{i(w(x)-w(t))}|\{|p(t)S_j(t) + R(t)T_j(t)|\}dt\right] + \left[\int_y^x |e^{i(w(x)-w(t))}|\{|p(t)S_j(t) + R(t)T_j(t)|\}dt\right]$$

From (16) and (17), we have

$$O\left[\int_0^y |e^{i(w(x)-2w(t))}|\{|p(t)| + |R(t)e^{i(w(t)-z(t))}|\}dt\right] + $$
$$+O\left[\int_y^x |e^{i(w(x)-2w(t))}|\{|p(t)| + |R(t)e^{i(w(t)-w(t))}|\}dt\right]$$

$$= O\left[e^{im(2w(y)-w(x))}\int_0^y \{|p(t)| + |R(t)|e^{im(z(t)-w(t))\}}dt\right] +$$
$$+ O\left[e^{im\,w(x)}\int_y^x \{|p(t)|dt\right] + O\left[e^{im\,z(x)}\int_y^x \{|R(t)|dt\right]$$

We now keep x fixed and choose y such that the last two integrals are very very small and after choosing y, the first term tends to zero as $x \to \infty$. Therefore, as $x \to \infty$.

$$O\left[\int_x^y + \int_y^x e\right] = 0\left(e^{im\,z(x)}\right).$$

Similarly, as $x \to \infty$

$$\int_x^\infty e^{-i(w(x)-w(t))}\{p(t)e^{-iw(t)} + R(t)e^{-iz(t)}\}dt$$
$$= O\left[\int_x^\infty e^{im(w(x)-w(t)}\{|p(t)|e^{im\,w(t)} + |R(t)|e^{im\,z(t)}\}dt\right]$$

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$$= O\left[e^{im\,w(x)}\int_x^\infty |p(t)|dt\right] + O\left[e^{im\,w(x)}\int_\alpha^\infty |R(t)|e^{im\,(z(t)-w(t))}dt\right]$$
$$= O\left(e^{im\,w(x)}\right).$$

Thus, as x → ∞

$S_j(x) = e^{-iw(x)}\left[B_{j1}(\lambda) + o(1)\right]$, (j = 1, 2) (18)

where

$$B_{j1}(\lambda) = \frac{1}{2}S_j(o) - \frac{1}{2i\sqrt{\lambda}}S_j(0) + \frac{1}{2i}\int_0^\infty e^{iw(t)}\{p(t)S_j(t) + R(t)T_j(t)\}dt$$

(j - 1, 2) (19)

Similarly as x → ∞

$T_j(x) = e^{-iz(x)}\left[B_{j2}(\lambda) + o(1)\right]$, (j = 1, 2) (20)

where

$$B_{j2}(\lambda) = \frac{1}{2}T_j(o) - \frac{1}{2i\sqrt{\lambda}}T_j(0) + \frac{1}{2i}\int_0^\infty e^{iz(t)}\{p(t)T_j(t) + R(t)S_j(t)\}dt$$

(j - 1, 2) (21)

Now if ϕ_j (x,λ) (j = 1,2) are boundary condition vectors, then by (1) we get

$$u_j(x,\lambda) = (\lambda - p(x))^{\frac{1}{4}}S_j(x)$$

and

$$v_j(x,\lambda) = (\lambda - q(x))^{\frac{1}{4}}T_j(x)$$

Hence by (2.4.10), we have, as x →∞.

$$u_j(x,\lambda) = \frac{e^{-iw(x)}[M_{j1}(\lambda) + o(1)]}{(\lambda - p(x))^{1/4}}$$

(j = 1,2) (22)

where

$$M_{j1}(\lambda) = \frac{1}{2}\lambda^{\frac{1}{4}}u_j(0) - \frac{1}{2i}\left\{\frac{u_j(0)}{\lambda^{\frac{1}{4}}} - \frac{1}{4}\frac{u_j(0)p'(0)}{\lambda^{\frac{5}{4}}}\right\} + \frac{1}{2i}\int_0^\infty e^{iw(t)}\{p(t)(\lambda - p(t))^{\frac{1}{4}}v_j(t)\}dt$$

(23)

and

$v_j(x,\lambda) = \frac{e^{-iz(x)}[M_{j2}(\lambda)+0(1)}{(\lambda - q(x))^{1/4}}$, $J = 1,2$ (24)

where

$$M_{j2}(\lambda) = \frac{1}{2}\lambda^{\frac{1}{4}}v_j(0) - \frac{1}{2i}\left\{\frac{v_j(0)}{\lambda^{\frac{1}{4}}} - \frac{1}{4}\frac{v_j(0)q'(0)}{\lambda^{\frac{5}{4}}}\right\} + \frac{1}{2i}\int_0^\infty e^{iz(t)}\{Q(t)(\lambda - q(t))^{\frac{1}{4}}v_j(t)\}dt$$

(25)

Similarly, if $\theta_k(x,\lambda) = \begin{pmatrix} x_k(x,\lambda) \\ y_k(x,\lambda) \end{pmatrix}$ $(mkl, 2) be\ the$

solutions of the system defined, we have as x →∞ .

$$x_k(x,\lambda) = \frac{e^{-iw(x)}[N_{k1}(\lambda) + 0(1)}{(\lambda - p(x))^{1/4}}$$

(k = 1, 2) (26)

where

$$N_{k1}(\lambda) = \frac{1}{2}\lambda^{\frac{1}{4}}x_k(0) - \frac{1}{2i}\left\{\frac{x_k'(0)}{\lambda^{\frac{1}{4}}} - \frac{1}{4}\frac{x_k(0)p'(0)}{\lambda^{\frac{5}{4}}}\right\} + \frac{1}{2i}\int_0^\infty e^{iw(t)}\{p(t)(\lambda - p(t))^{\frac{1}{4}}x_k(t) + R(t)(\lambda - q))^{\frac{1}{4}}y_k(t)\}dt$$

(27)

and

$$y_k(x,\lambda) = \frac{e^{-iz(x)}[N_{k2}(\lambda) + 0(1)}{(\lambda - q(x))^{1/4}}$$

(k = 1, 2) (28)

where

$$N_{k2}(\lambda) = \frac{1}{2}\lambda^{\frac{1}{4}}y_k(0) - \frac{1}{2i}\left\{\frac{y_k'(0)}{\lambda^{\frac{1}{4}}} - \frac{1}{4}\frac{y_k(0)q'(0)}{\lambda^{\frac{5}{4}}}\right\} + \frac{1}{2i}\int_0^\infty e^{iz(t)}\{Q(t)(\lambda - q(t))^{\frac{1}{4}}y_k(t) + R(t)(\lambda - p(t))^{\frac{1}{4}}x_k(t)\}dt$$

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(29)

By (26) and (27), (28), (29), we get

$$\psi_{j1}(x,\lambda) = \frac{e^{-iw(x)}}{(\lambda - p(x))^{\frac{1}{4}}}\left[N_{j1}(\lambda) + \sum_{s=1}^{2} m_{js}(\lambda)M_{s1}(\lambda) + 0(1)\right]$$

$$\psi_{j2}(x,\lambda) = \frac{e^{-iz(x)}}{(\lambda - q(x))^{\frac{1}{4}}}\left[N_{j2}(\lambda) + \sum_{s=1}^{2} m_{js}(\lambda)M_{s2}(\lambda) + 0(1)\right]$$

(j = 1,2)

since, $\psi_j(x,\lambda) = \begin{pmatrix}\psi_{j1}\\ \psi_{j2}\end{pmatrix}$(j = 1,2)are L^2 (0, ∞), there

$$N_{j1}(\lambda) + \sum_{s=1}^{2} m_{js}(\lambda)M_{s1}(\lambda) = 0$$

$$N_{j2}(\lambda) + \sum_{s=1}^{2} m_{js}(\lambda)M_{s2}(\lambda) = 0$$

Hence

$$m_{j1}(\lambda) = \frac{M_{kk}N_{j1} - M_{kl}N_{jk}}{M_{21}M_{12} - M_{11}M_{22}} \qquad (2.4.22)$$

(j = 1, 2; when l = 1, k = 2 and when l = 2, k = 1).

Since , are linearly independent, by (6) the denominator of (22) is not zero unless M_{21}, M_{12}, M_{11}, M_{22} are all zero. Bu arguments analogous to Titchmarah $\lambda^{\frac{1}{4}}M_{jk}(\lambda)(1 \le j,k \le 2)$ and $\lambda^{\frac{1}{4}}N_{jk}(\lambda)(1 \le j,k \le 2)$ are continuous up to the negative real axis and are real there, w(t), z(t), p(t), q(t) and R(t) are all being purely imaginary.

Exactly following the arguments of Tichmarsh it can be proved that $m_{rs}(\lambda)$ $(1 \le r, s \le 2)$ are meromorphic functions of λ. Hence the spectrum is discrete.

V. CONCLUSIONS

That for a sequence of values of b tending to infinity, just N of the numbers $\lambda_{n,b}$ lie in the interval $-R \le \lambda \le R$. Dnote these by $\mu_{1b}\ \mu_{2b} \cdots \mu_{Nb}$ in non-decreasing order. Then we can select a sub-sequence of values of b such that μ_{nb} tends to a limit, say μ_n for each n. Now we construct four functions.

$$f_{rs,b}(\lambda) = (\lambda - \mu_{1b})(\lambda - \mu_{2b}) \ldots \ldots (\lambda - \mu_{Nb})l_{rs}(\lambda)$$

$(1 \le r, s \le 2)$

The functions represented by (2.4.54) are thus regular for - R < re < R, and as b through the sub-sequence

$$f_{rs,b}(\lambda) \to f_{rs}(\lambda) = (\lambda - \mu_1)(\lambda - \mu_2) \ldots \ldots (\lambda - \mu_N)m_{rs}(\lambda) \qquad (1 \le r, s \le 2)$$

For any non real λ.

lrs(n) (1 < r, s < 2) are bounded and 0(1/υ) as υ→0 (See Bhagat [4, 4]. Hence it follows from Lemma 2.11 of Titchmarah that $f_{rs,b}$ (λ)$(1 \le r, s \le 2)$ are bounded if

$- R + 1 \le re\ \lambda \le R - 1$, $- R + 1 \le im\lambda \le R - 1$.

Thus $f_{rs,b}(\lambda) \to f_{rs,b}(\lambda)$ $(1 \le r, s \le 2)$ uniformly in any region interior to this, and so $f_{rs,b}(\lambda)$ $(1 \le r,s \le 2)$ are regular in such region. Thus mrs(λ) $(1 \le r,s \le 2)$ are regular except possibly for poles at $\mu_1, \mu_2, \ldots \ldots \mu_N$. Hence the result.

REFERENCES

[1] Khoff, G.D. and Langer, R.E. ‘ The boundary value problem and developments associated with a system of ordinary differential systems of the first order’. Proc. Amer. Acad. of Arts and Sciences, 58(1923): 51-128.

[2] Bhagat, B., ‘Eigenfunction expansions associated with a pair of second order differential equations’, Proc. National Inst. Sciences of India Vol. 35, A.No.1 (1969).

[3] Bhagat, B., ‘Some problems on a pair of singular second order differential equations’. Ibid 35A (1969): 232-44.

[4] Bhagat, B., ‘Ph.D. Thesis’. Patna University. 1966.

[5] Chakravarty, N.K. ‘Some problems in eigenfuction expansions I’ Quart. J. Math. (16) 62 (1965) 135-50.

[6] Conte, S.D. and Sangern. W.C., ‘ An asymptotic solution for a pair of singular first order equations’ Proc. Amer. Math. Boc. 4(1953): 696-702.

[7] Conte, S.D. and Sangern. W.C., ‘An expansion theorem for a pair of first order equation’ Canadian J. Math. 6 (1954): 554-60

[8] Choudhary, J. and Everitt. W.ll. ‘The spectrum of a fourth order differential operation’ Proc. Royal Soc. Of Edinburgh Sec. Vol. LXVIII Part III No. 13 (193-210).

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