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# Photogravitational Effect on the Sitnikov Five body Problem forming Kite Configuration

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**Abstract In this paper; the effect of photogravitation (radiation pressure) on the motion of the infinitesimal mass in the Sitnikov's Kite configuration has been studied, under the perturbation of photogravitation, the series solution of the equation of motion of the infinitesimal mass has been established by the iteration process of Green's function. By solving the variational equations the effect of photogravitation on the stability of libration points have been checked and found stable.**

***Keywords* —Sitnikov Problem, Kite Configuration, Photo – gravitation, Green's Function, Series Solution, Stability**

## I. INTRODUCTION

Pavanini [10] introduced the problem for the first time as the special case of the circular Restricted-three body problem (CR3BP) and MacMillan [8] expressed its solution in terms of Jacobi elliptic functions. After a long gap of almost half century Sitnikov [16] studied the problem in detail and proved the existence of oscillating motion of the restricted three-body problem. Stumpff [20] re discussed the above problem. Sitnikov's problem has further been studied by many authors. Perdios et al [12] have studied stability and bifurcation of Sitnikov motion. Liu and Sun [7] have studied the Sitnikov problem without taking the original differential equation and discovered an invariant set of hyperbolic solutions.

Hagel [5] has studied the problem by a new analytic approach. Faruque [3] has established the new analytical expression for the position of the infinitesimal body in the elliptic Sitnikov problem. Further by some author's chaotic motion also have been studied. Perdios [11] has studied the manifolds of families of three – dimensional periodic orbits in the three-body problem. Suraj, Hassan and Bhatnagar [21] have averaged the equation of motion of the Sitnikov restricted four-body problem under the gravitational forces and they further extended the problem when all the primaries are sources of radiation.

Shahbaz Ullah and Hassan [17] have studied the connection between three-body configuration and four-body configuration of the Sitnikov problem when one of the masses approaches zero: Circular case. Further Shahbaz Ullah and Hassan [19] have studied Sitnikov cyclic configuration of $(n+1)$ body problem and Sitnikov problem with kite configuration. Rahman and Hassan [13] have studied solution and stability of restricted three – body problem when the primaries are sources of radiation.

In the present work, we proposed to study the effect of photogravitation on the motion of infinitesimal mass in the Sitnikov five-body problem when the primaries form a kite configuration. Stability of libration points and Poincare section for periodicity has also been examined.

## II. EQUATIONS OF MOTION

Let the four radiating primaries $P_1, P_2, P_3$ and $P_4$ of masses $m_1, m_2, m_3$ and $m_4$ be moving on a common circular orbit with Centre at $O$ (Figure 1) such that $OP_i = R$ $(i = 1,2,3,4)$. $R$ is the radius of the common circular orbit and an infinitesimal body of mass $m$ at $P$ moving in space in the gravitational field four radiating primaries. Also it is supposed that the four primaries conserve their positions at the vertices of a central kite configuration under the restrictions $m_1 = m_3, m_1$ and $m_4$ are arbitrary, $r_{12} = r_{23}$ and $r_{14} = r_{34}$.

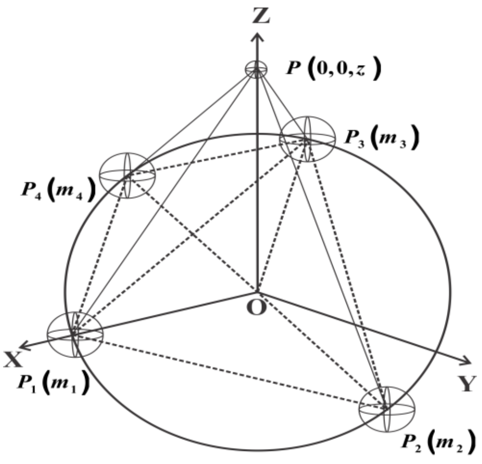


**Fig. 1 Sitnikov Five – body Problem forming Kite Configuration**

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The above restrictions represent $P_1P_2P_3$ as an equilateral triangle and $P_1P_3P_4$ as an isosceles triangle. If we consider all the above restrictions together, then additional restrictions automatically exists i.e., the motion of the infinitesimal mass becomes one dimensional along the line $OZ$ through $O$ and perpendicular to the plane of motion of the primaries. Considering the line $OZ$ as the $z$-axis, $OP_1$ along the $x-$axis and a line $OY$ perpendicular to the $x-$axis and lying on the plane motion as the $y-$axis, then

$$P_i = (R\cos\theta_i, R\sin\theta_i, 0),\ i = 1,2,3,4,$$

$$\theta_1 = 0,\ \theta_2 = 120^\circ,\ \theta_3 = 240^\circ,\ \theta_4 = 300^\circ,$$

$$P_1 \equiv (R, 0, 0),$$

$$P_2 \equiv \left(-\tfrac{1}{2}R, \tfrac{\sqrt{3}}{2}R, 0\right),$$

$$P_3 \equiv \left(-\tfrac{1}{2}R, -\tfrac{\sqrt{3}}{2}R, 0\right),$$

$$P_4 \equiv \left(\tfrac{1}{2}R, -\tfrac{\sqrt{3}}{2}R, 0\right).$$

Let at any time $t, P(0,0,z)$ be the position of the infinitesimal mass on the vertical $z-$axis and $\overrightarrow{OP_i} = \vec{r_i}$ then

$$\left.\begin{aligned}
\vec{r_1} &= \overrightarrow{PP_1} = R\hat{i} - z\hat{k},\\
\vec{r_2} &= \overrightarrow{PP_2} = -\frac{1}{2}R\hat{i} - R\hat{j} - z\hat{k},\\
\vec{r_3} &= \overrightarrow{PP_3} = -\frac{1}{2}R\hat{i} - \frac{\sqrt{3}}{2}R\hat{j} + z\hat{k},\\
\vec{r_4} &= \overrightarrow{PP_4} = \frac{1}{2}R\hat{i} - \frac{5}{2}R\hat{j} - z\hat{k},\\
|\vec{r_1}| &= |\vec{r_2}| = |\vec{r_3}| = |\vec{r_4}| = \sqrt{R^2 + z^2}.
\end{aligned}\right\} \quad (1)$$

Let $\hat{r_i}\ (i = 1,2,3,4)$ be the unit vectors along $\vec{r_i}$, then

$$\hat{r_i} = \frac{\vec{r_i}}{|r_i|},$$

$$\hat{r_1} = \frac{R\hat{i} - z\hat{k}}{\sqrt{R^2 + z^2}},$$

$$\hat{r_2} = \frac{-\frac{R}{2}\hat{i} + \frac{\sqrt{3}}{2}R\hat{j} - z\hat{k}}{\sqrt{R^2 + z^2}},$$

$$\hat{r_3} = \frac{-\frac{R}{2}\hat{i} - \frac{\sqrt{3}}{2}R\hat{j} - z\hat{k}}{\sqrt{R^2 + z^2}},$$

$$\hat{r_4} = \frac{\frac{R}{2}\hat{i} - \frac{\sqrt{3}}{2}R\hat{j} - z\hat{k}}{\sqrt{R^2 + z^2}}.$$

Further, let $\vec{F}_{g_i}$ be the gravitational force exerted by $m_i$ on $m$, then

$$\vec{F}_{g_i} = \frac{Gmm_i}{r_i^2}\hat{r_i}\ (i = 1,2,3,4) \text{ along } \overrightarrow{PP_i}.$$

Let $\vec{F}_{g_i}$ be radiation pressure of $m_i$ on $m$ along $\overrightarrow{P_iP}$, then total force exerted on $m$ by $m_i$ along $\overrightarrow{PP_i}$ is given by

$$= \vec{F}_{g_i} - \vec{F}_{p_i},$$

$$= \vec{F}_{g_i}(1 - q_i), \text{ where } q_i = \frac{\vec{F}_{p_i}}{\vec{F}_{g_i}} \square\ 1 \text{ i.e., } 0 < 1 - q_i < 1.$$

Total force exerted on $m$ by the four primaries is given by

$$\vec{F} = \sum_{i=1}^{4}(1 - q_i)\vec{F}_{g_i},$$

$$\vec{F} = \sum_{i=1}^{4}(1 - q_i)\frac{Gmm_i}{|r_i|^2}\hat{r_i},$$

$$\vec{F} = Gm\sum_{i=1}^{4}(1 - q_i)\frac{m_i}{r_i^3}\vec{r_i}.$$

Thus the equation of motion of the infinitesimal mass $m$ at $P(0,0,z)$ in gravitational field of four primaries $P_i\ (i = 1,2,3,4)$ in synodic (rotating) frame can be written as

$$m\left[\frac{\partial^2\vec{r}}{\partial t^2} + 2\vec{\omega}\times\vec{r} + \vec{\omega}\times(\vec{\omega}\times\vec{r})\right] = \vec{F},$$

where $\vec{r} = \overrightarrow{OP} = z\,k$, $\vec{\omega} = n\,k$.

$$\left.\begin{aligned}
\frac{\partial^2\vec{r}}{\partial t^2} + 2\vec{\omega}\times\vec{r} + \vec{\omega}\times(\vec{\omega}\times\vec{r}) = G\Bigg[&\frac{(1-q_1)m_1}{r_1^3}\left(R\,\hat{i} - z\,k\right)\\
&+ \frac{(1-q_2)m_2}{r_2^3}\left(-\tfrac{R}{2}\hat{i} + \tfrac{\sqrt{3}R}{2}j - z\,k\right)\\
&+ \frac{(1-q_2)m_3}{r_3^3}\left(-\tfrac{R}{2}\hat{i} - \tfrac{\sqrt{3}R}{2}j - z\,k\right)\\
&+ \frac{(1-q_2)m_4}{r_4^3}\left(\tfrac{R}{2}\hat{i} - \tfrac{\sqrt{3}R}{2}j - z\,k\right)\Bigg]
\end{aligned}\right\} \quad (2)$$

Taking the dot product of $\hat{k}$ on both sides of Equation (2), we get

$$\left.\begin{aligned}
\frac{d^2z}{dt^2} = -G\big[&(1-q_1)m_1 + (1-q_2)m_2 + (1-q_3)m_3\\
&+ (1-q_4)m_4\big]\frac{z}{(R^2+z^2)^{\frac{3}{2}}}.
\end{aligned}\right\} \quad (3)$$

Choosing unit of mass, unit of distance and unit of time in such a way that $2m_1 + m_2 + m_4 = 1, G = 1, R = 1, n = 1$. Then from Equation (3), we get the non – linear differential equation as

$$\frac{d^2z}{dt^2} + \frac{(1-q)z}{(z^2+1)^{\frac{3}{2}}} = 0, \quad (4)$$

where $q = \sum_{i=1}^{4} m_i q_i$.

To find the series solution by iteration of Green's function, we reduced the Equation (4) as

$$\frac{d^2z}{dt^2} + (1-q)z\left(1 - \frac{3}{2}z^2 + \ldots\right) = 0.$$

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As $z << 1$, so neglecting higher order terms of $z$ above the third and hence the above equation takes the form

$$\frac{d^2z}{dt^2}+(1-q)z-\frac{3}{2}(1-q)z^3=0 \tag{5}$$

## III. Series Solution by Green's Function

For the series solution by the iteration of Green's function, the Equation (5) can be written as

$$\frac{d^2z}{dt^2}+\eta_0^2 z=\left(\varepsilon z^2\right)z=-f\left(z\left(t\right)\right),\quad (\text{say}) \tag{6}$$

where $\eta_0^2=(1-q)$, $\varepsilon=\frac{3}{2}(1-p)$.

The general solution of the equation

$$\frac{d^2z}{dt^2}+\eta_0^2 z=0, \tag{7}$$

can be taken as

$$z=c\exp\left(i\eta_0 t\right),\quad \text{when}\left[z\left(0\right)=c\right]. \tag{8}$$

Suppose the solution of the Equation (6) is

$$z=c\exp\left(i\eta_0 t\right)+z^*\left(t\right). \tag{9}$$

Since the first term of the above equation satisfies the homogeneous Equation (7), hence we have

$$\frac{d^2z^*}{dt^2}+\eta_0^2 z^*=\left(\varepsilon z^2\right)z\equiv-f\left(z\left(t\right)\right), \tag{10}$$

where $z^*$ is the solution of the integral equation,

$$z^*(t)=\int G\left(t,\xi'\right)f\left(\xi'\right)d\xi', \tag{11}$$

where the Green's function $G$ is defined by the particular integral of the differential equation

$$\frac{d^2G}{dt^2}+\eta_0^2 G=-\delta\left(t-\xi'\right), \tag{12}$$

where the Dirac delta function $\delta(t-\xi')$ is defined as

$$\delta\left(t-\xi'\right)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp\left\{is\left(t-\xi'\right)\right\}ds. \tag{13}$$

The Green's function $G\left(t,\xi'\right)$ is the particular integral of Equation (6)

$$G\left(t,\xi'\right)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{1}{s^2-\eta_0^2}\exp\left\{is\left(t-\xi'\right)\right\}ds. \tag{14}$$

In the right hand side of Equation (14) $s=\pm\eta_0$ are the poles of order one, so by residue theorem,

$$G\left(t,\xi'\right)=\frac{i\exp\left\{i\eta_0\left(t-\xi'\right)\right\}}{2\eta_0}. \tag{15}$$

Now we find the solution of the Equation (10) by iteration process using the term $\left(\varepsilon z^2\right)$ as the iteration and using

$$f\left(t\right)=\int\delta\left(t-\xi'\right)f\left(\xi'\right)d\xi', \tag{16}$$

the first iteration is

$$z^*\left(\xi'\right)=-\varepsilon\int G\left(t,\xi'\right)\left(z\left(\xi'\right)\right)^2 z\left(\xi'\right)d\xi',$$

and the second iteration is

$$z^*(\xi'')=-\varepsilon\int G(\xi',\xi'')(z(\xi''))^2 z(\xi'')d\xi''.$$

Thus

$$z^*\left(\xi''\right)=\varepsilon^2\int\int G\left(t,\xi'\right)\left(z\left(\xi'\right)\right)^2 G\left(\xi',\xi''\right)\left(z\left(\xi''\right)\right)^2\times z\left(\xi''\right)d\xi''d\xi'.$$

Hence from the Equation (11), we have

$$z^*\left(t\right)=-\varepsilon\int G\left(t,\xi'\right)\left(z\left(\xi'\right)\right)^2 z\left(\xi'\right)d\xi' +\varepsilon^2\int\int G\left(t,\xi'\right)\left(z\left(\xi'\right)\right)^2 G\left(\xi',\xi''\right)\times\left(z\left(\xi''\right)\right)^2 z\left(\xi''\right)d\xi''d\xi'.$$

Finally from the Equation (9), we have

$$\left.\begin{aligned} z&=c\exp\left(t,\xi'\right)-\varepsilon\int G\left(t,\xi'\right)\left(z\left(\xi'\right)\right)^2 z\left(\xi'\right)d\xi' \\ &+\varepsilon^2\iint G\left(t,\xi'\right)\left(z\left(\xi'\right)\right)^2 G\left(\xi',\xi''\right)\left(z\left(\xi''\right)\right)^2\times \\ &z\left(\xi''\right)d\xi''d\xi'. \end{aligned}\right\} \tag{17}$$

Here in Equation (17), the iteration which gives the position of the infinitesimal mass**,** is given by the first approximation of the simple harmonic term of Equation (7) from $z\left(\xi''\right)$ to $z\left(\xi'\right)$ and then from $z\left(\xi'\right)$ to $z\left(t\right)$ at the time $\xi''$ and $\xi'$. The iterations are $\varepsilon z^2\left(\xi''\right)$, $\varepsilon z^2\left(\xi'\right)$ and the propagators are $G\left(\xi',\xi''\right), G\left(t,\xi'\right)$. To the first approximation, we use Equation (8) as

$$z(\xi')=c\exp\left(i\eta_0\xi'\right),$$

and Equation (17) as

$$z=c\exp\left(i\eta_0 t\right)+z_1+z_2,$$

where

$$\left.\begin{aligned} z_1&=-\varepsilon\int G\left(t,\xi'\right)\left(z\left(\xi'\right)\right)^2 z\left(\xi'\right)d\xi', \\ z_2&=\varepsilon^2\int\int G\left(t,\xi'\right)\left(z\left(\xi'\right)\right)^2 G\left(\xi',\xi''\right)\left(z\left(\xi''\right)\right)^2 z\left(\xi''\right)d\xi''d\xi' \end{aligned}\right\} \tag{18}$$

Now,

$$\begin{aligned} z_1&=-\varepsilon\int G\left(t,\xi'\right)\left(z\left(\xi'\right)\right)^3 d\xi', \\ &=-\varepsilon\int_0^t\frac{i\exp\left[i\eta_0\left(t-\xi'\right)\right]}{2\eta_0}c^3\exp\left(3i\eta_0\xi'\right)d\xi', \\ &=\frac{\varepsilon c^3}{4\eta_0^2}\left[\exp\left(i\eta_0 t\right)-\exp\left(3i\eta_0 t\right)\right], \end{aligned}$$

$$\left.\begin{aligned} z_1&=\frac{3}{8}c^3\left(\exp(i\eta_0 t)-\exp\left(3i\eta_0 t\right)\right)\quad \text{and} \\ z_2&=\left(\frac{3}{8}\right)^2 c^5\left[\exp\left(i\eta_0 t\right)-2\exp\left(3i\eta_0 t\right)+\exp\left(5i\eta_0 t\right)\right] \end{aligned}\right\} \tag{19}$$

Then, finally the solution of Equation (6) is

$$z=c\exp\left(i\eta_0 t\right)+z_1+z_2+\ldots$$

$$\left.\begin{aligned} z&=\left\{c+\frac{3}{8}c^3+\left(\frac{3}{8}\right)^2 c^5\right\}\exp\left(i\eta_0 t\right) \\ &-\left\{\frac{3}{8}c^3+2\left(\frac{3}{8}\right)^2 c^5\right\}\exp\left(3i\eta_0 t\right)+\left(\frac{3}{8}\right)^2 c^5\exp\left(5i\eta_0 t\right)+\ldots \end{aligned}\right\} \tag{20}$$


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is the required series solution by the iteration of Green's function.

## IV. STABILITY OF THE EQUILIBRIUM POINTS

Following Murray and Dermott (1999) let us check the Stability of the Sitnikov motion. We rewrite the general equations of motion given in Equation (5) as

$$\left.\begin{aligned} \ddot{x}-2n\dot{y} &= \Omega_x, \\ \ddot{y}+2n\dot{x} &= \Omega_y, \\ \ddot{z} &= \Omega_z, \end{aligned}\right\} \tag{21}$$

where the force function $\Omega$ is given by

$$\Omega = \frac{(1-q)}{(z^2+1)^{\frac{1}{2}}},$$

$$\Omega_z = -\frac{(1-q)z}{(z^2+1)^{\frac{3}{2}}} \tag{22}$$

and

$$\Omega_{zz} = -(1-q)+\frac{9}{2}(1-q)z^2-\frac{15}{2}(1-q)z^4. \tag{23}$$

The system of Equation (21) can be written as

$$\left.\begin{aligned} \ddot{x}-2n\dot{y} &= \Omega_x = f(x,y,z), \text{ say} \\ \ddot{y}-2n\dot{x} &= \Omega_y = g(x,y,z), \text{ say} \\ \ddot{z} &= \Omega_z = h(x,y,z). \text{ say} \end{aligned}\right\} \tag{24}$$

For stationary solution, $\Omega$ is a function of $z$ only, so there is no solution in the $xy$ – plane, clearly the solution lie on the $z$ – axis only. Let us denote the libration point as $P(0,0,z_0)$ then from Equation (26), we have

$$\left.\begin{aligned} \Omega_x^0 &= f(0,0,z_0)=0, \\ \Omega_y^0 &= g(0,0,z_0)=0, \\ \Omega_z^0 &= h(0,0,z_0) = -\frac{(1-p)z_0}{\left(z_0^2+\frac{1}{2}\right)^{\frac{3}{2}}}. \end{aligned}\right\} \tag{25}$$

where $\Omega_x^0, \Omega_y^0, \Omega_z^0$ are the values of $\Omega_x, \Omega_y, \Omega_z$ at the libration points.

We shall now communicate the small displacement $\xi, \eta, \zeta$ in the coordinate of $P$ such that

$x = 0+\xi, y = 0+\eta, z = z_0+\zeta.$

Equation (12) becomes

$$\begin{aligned} \ddot{\xi}-2n\dot{\eta} &= f(0+\xi, 0+\eta, z_0+\zeta), \\ \ddot{\eta}+2n\dot{\xi} &= g(0+\xi, 0+\eta, z_0+\zeta), \\ \ddot{\zeta} &= h(0+\xi, 0+\eta, z_0+\zeta) \qquad \text{where } \ddot{z}_0 = 0. \end{aligned}$$

Now applying the Taylor's theorem in the neighborhood of $(0,0,z_0)$, we get

$$\left.\begin{aligned} \ddot{\xi}-2n\dot{\eta} &= f(0,0,z_0)+\xi\left(\frac{\partial f}{\partial x}\right)_0+\eta\left(\frac{\partial f}{\partial y}\right)_0+\zeta\left(\frac{\partial f}{\partial z}\right)_0 \\ &\quad + \text{higher order infinitesimals}, \\ \ddot{\eta}+2n\dot{\xi} &= g(0,0,z_0)+\xi\left(\frac{\partial g}{\partial x}\right)_0+\eta\left(\frac{\partial g}{\partial y}\right)_0+\zeta\left(\frac{\partial g}{\partial z}\right)_0 \\ &\quad + \text{higher order infinitesimals}, \\ \ddot{\zeta} &= h(0,0,z_0)+\xi\left(\frac{\partial h}{\partial x}\right)_0+\eta\left(\frac{\partial h}{\partial y}\right)_0+\zeta\left(\frac{\partial h}{\partial z}\right)_0 \\ &\quad + \text{higher order infinitesimals}, \end{aligned}\right\} \tag{26}$$

By using Equation (5), the system of Equation (26) reduced to

$$\left.\begin{aligned} \ddot{\xi}-2n\dot{\eta} &= \xi\frac{\partial}{\partial x}\left(\frac{\partial\Omega}{\partial x}\right)_0+\eta\frac{\partial}{\partial y}\left(\frac{\partial\Omega}{\partial x}\right)_0+\zeta\frac{\partial}{\partial z}\left(\frac{\partial\Omega}{\partial x}\right)_0 \\ &\quad + \text{higher order infinitesimals}, \\ \ddot{\eta}+2n\dot{\xi} &= \xi\frac{\partial}{\partial x}\left(\frac{\partial\Omega}{\partial y}\right)_0+\eta\frac{\partial}{\partial y}\left(\frac{\partial\Omega}{\partial y}\right)_0+\zeta\frac{\partial}{\partial z}\left(\frac{\partial\Omega}{\partial y}\right)_0 \\ &\quad + \text{higher order infinitesimals}, \\ \ddot{\zeta} &= \xi\frac{\partial}{\partial x}\left(\frac{\partial\Omega}{\partial z}\right)_0+\eta\frac{\partial}{\partial y}\left(\frac{\partial\Omega}{\partial z}\right)_0+\zeta\frac{\partial}{\partial z}\left(\frac{\partial\Omega}{\partial z}\right)_0 \\ &\quad + \text{higher order infinitesimals}, \end{aligned}\right\} \tag{27}$$

where $f = \frac{\partial\Omega}{\partial x}$, $g = \frac{\partial\Omega}{\partial y}$ and $h = \frac{\partial\Omega}{\partial z}$.

Neglecting the higher order terms of $\xi, \eta$ and $\zeta$, we get the new variational equations

$$\left.\begin{aligned} \ddot{\xi}-2n\dot{\eta} &= \xi\Omega_{xx}^0+\eta\Omega_{yx}^0+\zeta\Omega_{zx}^0, \\ \ddot{\eta}+2n\dot{\xi} &= \xi\Omega_{xy}^0+\eta\Omega_{yy}^0+\zeta\Omega_{zy}^0, \\ \ddot{\zeta} &= \xi\Omega_{xz}^0+\eta\Omega_{yz}^0+\zeta\Omega_{zz}^0. \end{aligned}\right\} \tag{28}$$

where $\Omega_{xx}^0, \Omega_{xy}^0, \Omega_{xz}^0 \ldots$ represent the second order derivatives of $\Omega$ at the libration points.

The system of Equation (28) can be written in the form of a single matrix equation as

$$\dot{X} = AX, \tag{29}$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \Omega_{xx}^0 & \Omega_{yx}^0 & \Omega_{zx}^0 & 0 & 2n & 0 \\ \Omega_{xy}^0 & \Omega_{yy}^0 & \Omega_{zy}^0 & -2n & 0 & 0 \\ \Omega_{xz}^0 & \Omega_{yz}^0 & \Omega_{zz}^0 & 0 & 0 & 0 \end{bmatrix} \text{ and } X = \begin{bmatrix} \xi \\ \eta \\ \zeta \\ \dot{\xi} \\ \dot{\eta} \\ \dot{\zeta} \end{bmatrix}$$

If any matrix $X$ satisfy the equation

$$AX = \lambda X, \tag{30}$$

Then $X$ is said to be an Eigen vector of the matrix $A$ and scalar $\lambda$ is its corresponding Eigen value. If $A$ is thought of as a transformation matrix, then the result of applying $A$ to the particular vector $X$ satisfying Equation (30) is to produce a vector in the same direction as $X$ but

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of a different magnitude. Now the Equation (30) can be written as

$$(A-\lambda I)X = 0,$$

The set of six simultaneous linear equations in six unknowns $\xi, \eta, \zeta, \dot{\xi}, \dot{\eta}, \dot{\zeta}$ will have non – trivial solutions provided the determinant of the matrix $(A-\lambda I)$ vanishes.

i.e., $|A-\lambda I| = 0.$ (31)

Now, the non – trivial solution of Equation (31) will be stable if they are periodic and the solutions are periodic, if the Eigen values of the matrix $A$ are either zero or imaginary. The Equation (31) yields

$$\lambda^2\left(\lambda^2+4n^2\right)\left(\lambda^2-\Omega_{zz}^0\right)=0. \quad (32)$$

The Equation (32) is a polynomial equation of degree six in $\lambda$, so there will be three roots in $\lambda^2$ corresponding to the three factors of Equation (32). The conditions for stable solutions are

$$\lambda_i^2 \le 0 \ (i=1,2,3),$$

where $\lambda_1^2 = 0, \lambda_2^2 = -4n^2, \lambda_3^2 = \Omega_{zz}^0$ are three roots of Equation(32).

Since $\lambda_1^2 = 0$ hence $\lambda_{11} = \lambda_{12} = 0$. When $\lambda_2^2 = -4n^2 < 0$ then $\lambda_2 = \pm 2ni$. i.e., $\lambda_{21} = 2ni$ and $\lambda_{22} = -2ni$. When $\lambda_3^2 = \Omega_{zz}^0$, since $z_0 << 1$ hence the quantity containing higher power of $z_0$ above the second must be neglected, therefore

$$\lambda_3^2 = \Omega_{zz}^0 = -\left(\frac{1-q}{2}\right)\left(2+9z_0^2\right) < 0,$$

$$\lambda_{31} = i\left\{\left(\frac{1-q}{2}\right)\left(2+9z_0^2\right)\right\}^{\frac{1}{2}}$$

and $\lambda_{32} = -i\left\{\left(\frac{1-q}{2}\right)\left(2+9z_0^2\right)\right\}^{\frac{1}{2}}$. (imaginary)

Thus all the six roots of the characteristic equations are $0, 0$

$$2ni, -2ni, i\left\{\left(\frac{1-q}{2}\right)\left(2+9z_0^2\right)\right\}^{\frac{1}{2}}, -i\left\{\left(\frac{1-q}{2}\right)\left(2+9z_0^2\right)\right\}^{\frac{1}{2}},$$

i.e., they are either zero or imaginary and hence the libration points of the Sitnikov five body problem are stable.

## V. Discussions and Conclusion

In section (I), we have introduced the contribution of successive authors of the field of Sitnikov motion with extension. In section (II) we have derived the equation of motion of the infinitesimal mass moving in the gravitational field of four radiating primaries of masses $m_1, m_2, m_3$ and $m_4$ along the Z axis which passes through the Centre of mass of the four primaries.

To maintain the Kite configuration, the restriction on the masses of the primaries are,
(i) $m_1 = m_3$,
(ii) $m_2$ & $m_4$ are arbitrary,
(iii) $m_1 + m_2 + m_3 + m_4 = 1$.

With the above restrictions, we derive the linear equation of motion of the infinitesimal mass in a standard form suitable for iteration process of Green’s function. In section (III) than a series solution has been obtained, with no effect of photogravitation. In section (IV) the variational equations have been established by Taylor’s theorem in the neighborhood of libration points. By discussing the nature of roots of the characteristic equation, we have claim that, all the roots are either Zero or purely imaginary and hence the libration point are stable.

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