International Journal for Research in Engineering Application & Management (IJREAM)
ISSN : 2454-9150 Vol-05, Issue-03, June 2019

On The Generalized Leguerre Polynomial With Wright's Generalized Hypergeometric Function And Applications

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Abstract - The aim of this paper is to evaluate generalized Leguerre polynomial with Wright's generalized hypergeometric function defined by Dotsenko [1, 2]. The author has given two applications of generalized Leguerre polynomial with Wright's generalized hypergeometric function by connecting this, first with the Weyl integral and second is with Riemann-Liouville type of fractional derivative.The results obtained are basic in nature and are likely to find useful applications.

(Mathematics Subject Classification: 33C20, 33E20)

Key words: Generalized Leguerre polynomial, Wright's Generalized Hypergeometric Function, Weyl Integral, Riemann-Liouville type fractional derivative.

I. Introduction

Generalized Wright's function ${}_2R_1\left(a,b;c,w;\mu;z\right)$ defined by Dotsenko [1, 2] hs been denoted as

$$ {}_2R_1(a,b;c,w;\mu;z)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} $$

$$ \times\sum_{k=0}^{\infty}\frac{\Gamma(a+k)\Gamma\left(b+k\frac{w}{\mu}\right)}{\Gamma\left(c+k\frac{w}{\mu}\right)}\frac{z^k}{k!} $$

$$ =\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\,{}_2\psi_1\left[z\left|\begin{matrix}(a,1),\left(b,\frac{w}{\mu}\right)\\ \left(c,\frac{w}{\mu}\right)\end{matrix}\right.\right] \tag{1.1} $$

Provided $\mathrm{Re}(c)>0,\mathrm{Re}(b)>0,\mathrm{Re}\left(\frac{w}{k}\right)>0$.

Virchenko et. al. [6] defined the Wright type hypergeometric function by taking $\frac{w}{k}=\tau>0$ in (1.1) as

$$ {}_2R_1^{\tau}(z)={}_2R_1(a,b;c,w;\mu;z) $$

$$ =\frac{\Gamma(c)}{\Gamma(b)}\sum_{k=0}^{\infty}\frac{(a)_k\Gamma\left(b+k\frac{w}{\mu}\right)}{\Gamma\left(c+k\frac{w}{\mu}\right)}\frac{z^k}{k!},\tau>0,|z|<1 $$

(1.2)

If $\tau=1$, then (1.2) reduces to a Gauss's hypergeometric function.

The generalized Leguerre function is defined in the form:

$$ L_n^{\alpha}(x)=\frac{(1+\alpha)_n}{n!}\,{}_1F_1[-n;1+\alpha;x] \tag{1.3} $$

II. Generalized Leguerre Transform of Wright's Generalized Hypergeometric Function

If $a,b,c,p,\gamma\in C;\mathrm{Re}(a)>0,\mathrm{Re}(b)>0,$

$\mathrm{Re}(c)>0,\mathrm{Re}(\delta)>0$ and $\frac{w}{k}\in N$, then

$$ \int_0^{\infty}x^{\gamma}e^{-x}L_n^{\alpha}(x)\,{}_2R_1(a,b;c,w;\mu;px^{\delta})dx $$

$$ =\frac{\Gamma(c)2^{\rho-1}}{\Gamma(a)\Gamma(b)}\frac{(1+\alpha)_n}{n!}\,{}_3\psi_1\left[p\left|\begin{matrix}(a,1),\left(b,\frac{w}{\mu}\right),(\gamma+\delta,1)\\ \left(c,\frac{w}{\mu}\right)\end{matrix}\right.\right] $$

$$ {}_2R_1[-n,\gamma+\delta k+1;1+\alpha;1] \tag{2.1} $$

Proof: $\int_0^{\infty}x^{\gamma}e^{-x}L_n^{\alpha}(x)\,{}_2R_1(a,b;c,w;\mu;px^{\delta})dx$

$$ =\frac{\Gamma(c)2^{\rho-1}}{\Gamma(a)\Gamma(b)}\frac{(1+\alpha)_n}{n!}\,{}_1F_1[-n;1+\alpha;1] $$

 DOI : 10.35291/2454-9150.2019.0225

ISSN : 2454-9150 Vol-05, Issue-03, June 2019

$$\times\sum_{k=0}^{\infty}\frac{\Gamma(a+k)\Gamma\left(b+k\frac{w}{\mu}\right)}{\Gamma\left(c+k\frac{w}{\mu}\right)}\int_0^{\infty}x^{\gamma+m+\delta k}e^{-k}dx$$

By using definition of generalized hypergeometric function ${}_2F_1(a,b;c;x)$ and Wright's generalized hypergeometric function ${}_2R_1(a,b;c,w;\mu;x)$, we arrive at the desired result.

III. Applications

The Weyl integral ([3], p.91) of $f(x)$ of order α, denoted by ${}_xW_\infty^\alpha$, is defined by

$$\left({}_xW_\infty^\alpha f\right)(x)=\left({}_xI_\infty^\alpha f\right)(x)=\left(I_-^\alpha f\right)(x)$$

$$=\frac{1}{\Gamma(\alpha)}\int_x^{\infty}(x-t)^{\alpha-1}f(t)dt,-\infty,x<\infty \qquad (3.1)$$

Where $\alpha\in C,\mathrm{Re}(\alpha)>0$.

The Weyl Integral of Generalized Leguerre Transform in Association with Wright's Generalized Hypergeometric Function

The main integral (2.1) can be rewritten as the following Weyl integral formula:

$$\left({}_0W_\infty^\rho e^{-x}L_n^\alpha(x)\,{}_2R_1(a,b;c,w;\mu;px^\delta)\right)(x)$$

$$=$$

$$\frac{1}{\Gamma(\gamma+1)}\int_0^{\infty}x^\gamma e^{-x}L_n^\alpha(x)\,{}_2R_1(a,b;c,w;\mu;px^\delta)dx$$

$$=\frac{1}{\Gamma(\gamma+1)}\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\frac{(1+\alpha)_n}{n!}$$

$$\times{}_3\psi_1\left[p\left|\begin{matrix}(a,1),\left(b,\frac{w}{\mu}\right),(\gamma+\delta,1)\\ \left(c,\frac{w}{\mu}\right)\end{matrix}\right.\right]{}_2R_1[-n,\gamma+\delta k+2;1+\alpha;1]$$

(3.2)

Provided $a,b,c,p,\gamma\in C;\mathrm{Re}(a)>0,\mathrm{Re}(b)>0,$

$\mathrm{Re}(c)>0,\mathrm{Re}(\delta)>0$ and $\frac{w}{k}\in N$,

Fractional Derivatives

Following Miller ([4],p.82), let $g\in A$ (Where A is a class of good functions). Then

$${}_zD_\infty^q g(z)=\frac{(-1)^q}{\Gamma(-q)}\int_z^{\infty}(u-z)^{-q-1}g(u)du,\ for q<0$$

(3.3)

For $q\geq 0$

$${}_zD_\infty^q g(z)=\frac{d^r}{dz^r}\left({}_zD_\infty^{q-r}g(z)\right) \qquad (3.4)$$

r being a positive integer such that $r>q$.

Fractional Derivatives of Generalized Leguerre Transform in Association with Wright's Generalized Hypergeometric Function

The main integral (2.1) can be rewritten as the following Riemann-Liouville fractional derivative formula:

$${}_0D_\infty^{-\rho}\left(e^{-x}L_n^\alpha(x)\,{}_2R_1(a,b;c,w;\mu;px^\delta)\right)$$

$$=\frac{(-1)^{\gamma+1}}{\Gamma(\gamma+1)}\int_0^{\infty}x^\gamma e^{-x}L_n^\alpha(x)\,{}_2R_1(a,b;c,w;\mu;px^\delta)dx$$

$$=\frac{(-1)^{\gamma+1}}{\Gamma(\gamma+1)}\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\frac{(1+\alpha)_n}{n!}.$$

$$\times{}_3\psi_1\left[p\left|\begin{matrix}(a,1),\left(b,\frac{w}{\mu}\right),(\gamma+\delta,1)\\ \left(c,\frac{w}{\mu}\right)\end{matrix}\right.\right]{}_2R_1[-n,\gamma+\delta k+2;1+\alpha;1]$$

(3.5)

It is being assumed that the conditions given in (2.1) and (3.3) are satisfied.

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DOI : 10.35291/2454-9150.2019.0225