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Fixed point theorem in 2 Banach Space and in Generalized 2 Banach Space for Contraction Mapping

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Abstract - The Theory of fixed point is applied in many fields of mathematics as well as in other studies. This is the main reason for producing new results as well as here prove fixed point theorems of contraction mapping on 2 Banach space and generalized 2 Banach space. In this paper are extension of fixed point theorem in 2 Banach space [5] and also extended to generalized 2 Banach Space [1]

***Keywords*: 2 normed space, 2 Banach space, fixed point, contraction mapping, Generalized 2 normed space, Generalized 2 Banach space.**

I. INTRODUCTION

In Mathematics a fixed point theorem is a result saying that a function F will have at least one fixed point (a point x for F(x)=x) under some conditions on F that can be stated in general terms.

The concept of 2 banach space firsly introduced by Gahler 1964. This space was subsequently been studied by Kir. M, Kiziltunc. H [6] and Malceski. R, Anevska. K [7]. Further Ramakant. B, Balaji .R.W, Basant .K. S[1] and Tiwari. S. K, Ranu. M [8] are prove the fixed point theorem in generalized Banach Space.

Kannan.R [3] and Chatterea [4] gave result in fixed point theorem in complete metric space and also proved fixed point theory for multivalued generalized contraction on a set with two b-matrices by monical.B [2] . Elkoch.V and Marhrani.E.M [9] solved result in fixed point theorem in generalized metric space.

Malceski. R and Ibrahimi. A [5] are proved that (L,||.,.||) be a 2-Banach space and S: L→L be mapping,so that

$\|Sx-Sy,z\|\leq\alpha(\|x-Sx,z\|+\|y-Sy,z\|)+\gamma(\|x-y,z\|)$ (or)

$\|Sx-Sy,z\|\leq\alpha(\|x-Sy,z\|+\|y-Sx,z\|)+\gamma(\|x-y,z\|)$

for all x,y,z ε L and $\alpha>0, \gamma\geq 0$ and $0<2\alpha+\gamma<1$ thenS has a unique fixed point in L. They are proved results in 2- Banach Space for contraction mapping. Malceski,A, Malceski.S, Anevska. K and Malceski,R, are proved result in New Extension of Kannan and Chatterjea Fixed point Theorems on Complete Metric Spaces [10],

In this paper I discussed some basic definitions in vector 2 normed space, generalized 2 nomed linear space and lemma. My aim of this paper is to show that fixed point theorems in 2 Banach Space and in generalized 2- Banach Space for contraction mapping. This paper is divided into two parts

Section I A Fixed point theorem in 2 Banach Space for contraction mapping.

Section II A Fixed point theorem in Generalized 2 Banach Space for contraction mapping.

II. PRELIMINARIES

Definition 2.1

Let M be a real vector space with dim M ≥2. 2-norm is a real function ||.,.||: MXM → [0,∞) such that

i)||x,y|| ≥ 0 , for all x,y ε M and ||x,y|| = 0 iff the set {x,y} is linearly dependent,

ii) ||x,y|| = ||y,x||, for all x,y ε M

iii) ||αx,y|| ≤ |α| ||y,x|| for all x,y ε M and α ε R

iv) ||x+y , z|| ≤ ||x,z|| + ||y,z|| for all x,y,z ε M

The ordered pair (M,||.,.||) is called a 2 normed space.

Definition 2.2

The sequence $\{x_n\}_{n=1}^{\infty}$ into vector 2 normed space is called as convergent if there exist x ε M such that $\lim_{n\to\infty}\|x_n-x,y\|=0$, for every y ε M. vector x ε M is called as bound of the sequence $\{x_n\}_{n=1}^{\infty}$ and we note $\lim_{n\to\infty} x_n = x \, or \, x_n \to x, n\to\infty$

Definition 2.3

Let M be a 2- normed space. The mapping S:M→M is said continuous mapping in x if for any sequence $\{x_n\}$ in M such that $x_n \to x \; for \, n\to\infty$ as the following holds true $Sx_n \to Sx$,n→∞.

Definition 2.4

The sequence $\{x_n\}_{n=1}^{\infty}$ in a vector 2 normed space M is Cauchy sequence if $\lim_{n\to\infty}\|x_n-x_m,y\|=0$,

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for every y ε M. For 2 normed space M, we say that is n-Banach space if every Cauchy sequence is convergent.

Definition 2.5

Let (M, ||.,.||) be a real vector 2 normed space. The mapping S:M→M is contraction if it exists h ε [0,1) so that

||Sx-Sy,z|| ≤ h||x-y,z||,for all x,y,z ε M holds true.

Definition 2.6

If M is a linear space having s(>=) ε R, let || .,. || denotes a function from linear space M that satisfies the following axioms such that for all x,y,z ε M

i) ||x,y|| = 0 iff x & y are linearly dependent

ii)||x,y|| = ||y,x||

iii)||x,βy||≤|β|||y,x||

iv)||x,y+z||≤s[||x,y|+||x,z||

||x|| is called norm of x and (M, ||.||) is called generalized 2 - normed linear space.If for s=1, it reduced to standard 2 -normed linear space.

Definition 2.7

A linear generalized 2-normed space in which every sequence is convergent is called generalized 2-Banach space.

Definition 2.8

Let (M, ||.||) be a generalized 2-normed linear space then the sequence $\{x_n\}$ in M is called

i)cauchy sequence iff for all ε>0, there exist n(∈)εN such that for each m,n ≥n(∈) we have $||x_n - x_m ,y|| < \in$

ii) convergent sequence iff there exist x ε M such that for all ε>0 there exist n(∈)εN such that for every n ≥n(∈)

we have $||x_n - x , y|| < \in$

Definition 2.9

The generalized 2-Banach space is complete if every Cauchy sequence converges.

Lemma 2.10:

Suppose (M,||.||) be a generalized Banach space and $\{y_n\}$ be a sequence in M such that

$|| y_{n+1} - y_{n+2}|| \leq h ||y_n - y_{n+1}||$, n=0,1,2….

Where 0 ≤ h < 1 then the sequence is Cauchy sequence in M provided sh < 1

Now in Section I, I will find a fixed point theorem in 2 Banach Space for contraction mapping.

III. FIXED POINT THEOREM IN 2 BANACH SPACE FOR CONTRACTION MAPPING

Theorem 3.1

Let (M,|| .,.||) be a 2 Banach Space and $T: M \to M$ be mapping so that

$$\|T(x)-T(y),z\| \leq a\|y-T(y),z\| + b\|y-T(x),z\| + c\|x-y\| \text{ ...... (1)}$$

for all x,y,z $\in M$ where a,b and c are non negative real number and satisfy 0 < a+2b+c < 1 and b+c < 1 , then T has a unique fixed point in M.

Proof

Let x_0 be any point on M and the sequence $\{x_n\}$ be defined as the following $x_{n+1} = Tx_n$, n=0,1,2…. the inequality (1) implies that

$$\|x_{n+1}-x_n,z\| = \|Tx_n - Tx_{n-1}\|$$

$$\leq a\|x_{n-1}-Tx_{n-1},z\| + b\|x_{n-1}-Tx_n\| + c\|x_n - x_{n-1}\|$$

$$\leq a\|x_{n-1}-x_n,z\| + b\|x_{n-1}-x_{n+1}\| + c\|x_n - x_{n-1}\|$$

$$\leq a\|x_{n-1}-x_n,z\| + b\|x_{n-1}-x_n\| + b\|x_n - x_{n+1}\| + c\|x_n - x_{n-1}\|$$

$$(1-b)\|x_n - x_{n+1}\| \leq (a+b+c)\|x_n - x_{n-1},z\|$$

$$\|x_n - x_{n+1},z\| \leq \frac{a+b+c}{1-b}\|x_n - x_{n-1},z\|$$

Holds true for each n=0,1,2,…. For each z $\in M$, the condition of the theorem implies that

$$h = \frac{a+b+c}{1-b} < 1$$

$$\|x_n - x_{n+1},z\| \leq h\|x_n - x_{n-1},z\| \text{ ..... (2)}$$

Holds true for each non negative, n=0,1,2,…. and each z $\in M$, Further (2) implies that

$$\|x_n - x_{n+1},z\| \leq h^n\|x_1 - x_0,z\| \text{ ..... (3)}$$

This implies that T is contraction mapping.

Holds true for each n=0,1,2…. and z $\in M$.

Now to show that $\{x_n\}$ is Cauchy sequence in M.

Using the properties of 2 norm and (3) we get

$$\|x_{m+k} - x_k,z\| \leq \|x_{m+k} - x_{m+k-1},z\| + \|x_{m+k-1} - x_{m+k-2},z\| + .$$

$$+......+\|x_{k+1} - x_k,z\|$$

$$\leq (\lambda^{m+k-1} + \lambda^{m+k-2} + ...\lambda^k)\|x_1 - x_0,z\|$$

$$\leq \lambda^k(\lambda^{m-1} + \lambda^{m-2} + ...1)\|x_1 - x_0,z\|$$

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$$\|x_{m+k} - x_k\| \le \frac{\lambda^k}{1-\lambda}\|x_1 - x_0, z\|$$

Holds true for all m , k ε M and z ε M

Since h < 1, Taking m, k $\to \infty$, $\lim_{m,k \to \infty} \|x_{m+k} - x_k, z\| = 0$

Holds true for all m, k ε M and z ε M

The sequence $\{x_n\}$ is Cauchy sequence in M.

Since M is 2 Banach Space, therefore the sequence $\{x_n\}$ is convergent sequence to u,

(i.e) $u \in M$ so that $\lim_{n \to \infty} \{x_n\} = u$

Now to show that u is fixed point on T.

$$\|u - Tu, z\| \le \|u - x_{n+1}, z\| + \|x_{n+1} - Tu, z\|$$

$$\le \|u - x_{n+1}, z\| + \|Tx_n - Tu, z\|$$

$$= \|u - x_{n+1}, z\| + a\|u - Tu, z\| + b\|u - Tx_n, z\|$$

$$+ c\|x_n - u, z\|$$

Taking limit n→∞, we get

$$\|u - Tu, z\| \le a\|u - Tu, z\|$$

$\|u - Tu, z\| = 0$, Since a < 1

Tu = u

(i.e) u is a fixed point on T.

We will prove that T has a unique fixed point on M.

Let u,w ε M be two distinct fixed points on T,

such that Tu = u and Tv = v then

$$\|u - Tu, z\| = \|Tu - Tv, z\|$$

$$\le a\|v - Tv, z\| + b\|v - Tu, z\| + c\|u - v, z\|$$

$$\le (b + c)\|u - v, z\|$$

Holds true for each z ε M.

$\|u - v, z\| = 0$ since b+c <1

u=v

T has a unique fixed point on M.

Now in Section II, I will find a fixed point theorem in generalized 2 Banach Space for Contraction mapping.

IV. FIXED POINT THEOREM IN GENERALIZED 2 BANACH SPACE FOR CONTRACTION MAPPING

Theorem 4.1

Let M be a Generalized 2 Banach Space with || .|| and $T : M \to M$ mapping

$$\|T(x) - T(y)\| \le a\|y - T(y), z\| + b\|y - T(x), z\|$$

$$+ c\|x - y\| \qquad \ldots\ldots(5)$$

for all x, y, z $\in M$ where a,b and c are non negative real number and satisfy b+ s (a+b+c) < 1 for s $\ge$ 1 and b+c <1 ,then T has a unique fixed point.

Proof

Let x_0 ε M be a sequence in M such that

$x_n = Tx_{n-1} = T^n x_0$, then

$$\|x_{n+1} - x_n, z\| = \|Tx_n - Tx_{n-1}, z\|$$

$$\le a\|x_{n-1} - Tx_{n-1}, z\| + b\|x_{n-1} - Tx_n, z\| + c\|x_n - x_{n-1}, z\|$$

$$= a\|x_{n-1} - x_n, z\| + b\|x_{n-1} - x_{n+1}, z\| + c\|x_n - x_{n-1}, z\|$$

$$= a\|x_{n-1} - x_n, z\| + b\|x_{n-1} - x_n, z\|$$

$$+ b\|x_n - x_{n+1}, z\| + c\|x_n - x_{n-1}, z\|$$

$$(1 - b)\|x_n - x_{n+1}, z\| \le (a + b + c)\|x_n - x_{n-1}, z\|$$

$$\|x_n - x_{n+1}, z\| \le \frac{(a + b + c)}{1 - b}\|x_n - x_{n-1}, z\|$$

$= h\|x_n - x_{n-1}, z\|$ where $h = \frac{a + b + c}{1 - b}$

Continuing this process we can easily say that

$$\|x_n - x_{n+1}, z\| \le h^n\|x_1 - x_0, z\|$$

This implies that T is contraction mapping.

Now it is to show that $\{x_n\}$ is Cauchy sequence in M.

Let m, n > 0 with m > n then from (1) we have,

$$\|x_n - x_m, z\| \le s\{\|x_n - x_{n+1}, z\| + \|x_{n+1} - x_m, z\|\}$$

$$\le s\|x_n - x_{n+1}, z\| + s^2\|x_{n+1} - x_{n+2}, z\|$$

$$+ s^3\|x_{n+2} - x_{n+3}, z\| + \ldots.$$

$$\le sh^n\|x_0 - x_1, z\| + s^2h^{n+1}\|x_0 - x_1, z\|$$

$$+ s^3h^{n+2}\|x_0 - x_1, z\| + ..$$

$$\le sh^n\|x_0 - x_1, z\|[1 + sh + (sh)^2 + \ldots..]$$

$$= \frac{sh^n}{1 - sh}\|x_0 - x_1, z\|$$

Now using the lemma 2.10 and taking limit n→∞, We get $\lim_{n \to \infty} \|x_n - x_m, z\| = 0$

therefore $\{x_n\}$ is a Cauchy sequence in M.

Since M is complete ,

we consider that $\{x_n\}$ converges to x*.

Now we show that x* is fixed point of T.

$$\|x^* - Tx^*, z\| \le s[\|x^* - x_n, z\| + \|x_n - Tx^*, z\|]$$

$$\le s[\|x^* - x_n, z\| + \|Tx_{n-1} - Tx^*, z\|]$$

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$$= s[\|x^* - x_n, z\| + a\|x^* - Tx^*, z\| + b\|x^* - Tx_{n-1}, z\| + c\|x_{n-1} - x^*, z\|]$$

$$(1 - as)\|x^* - Tx^*, z\| \le s[\|x^* - x_n, z\| + b\|x^* - x_n, z\| + c\|x_{n-1} - x^*, z\|]$$

$$\|x^* - Tx^*, z\| \le \frac{s}{1 - as}[\|x^* - x_n, z\| + b\|x^* - x_n, z\| + c\|x_{n-1} - x^*, z\|]$$

Taking limit n→∞ we get

$$\lim_{n \to \infty} \|x^* - Tx^*, z\| = 0$$

$$Tx^* = x^*$$

x^* is fixed point of T

Now for uniqueness of fixed point ,

Let x and y be two fixed point of T such that x=Tx and y=Ty then

$$\|x - y, z\| = \|Tx - Ty, z\|$$

$$\le a\|y - Ty, z\| + b\|y - Tx, z\| + c\|x - y, z\|$$

$$\|x - y, z\| \le (b + c)\|x - y, z\|$$

We get $\|x - y, z\| = 0$, since b+c <1

therefore x = y

This complete the theorem.

V. CONCLUSIONS

The aim of this paper are proven two theorems which are fixed point theorem on 2 Banach Space and fixed point theorem on generalized 2 Banach space for contraction mapping.

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