Solution of Instability Phenomenon in Homogeneous Porous Media with and Without Inclination by Differential Transform Method

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Abstract Over the lifetime of the reservoir, the pressure will fall and, at some point, there'll be scarce amount of underground pressure to force the oil to the surface. Secondary oil recovery techniques increase the reservoir pressure by water injection, which pushes oil towards the well. During this study, instability phenomenon has been observed when the water is injected in an inclined oil formatted area due to viscosity difference between water and oil. The non-linear partial differential equation for the instability phenomenon has been obtained. The differential transform method has been applied to the governing equation by using appropriate initial conditions. We've got compared the numerical value of saturation of injected water with and without inclination, too. The numerical value and graphical presentation are given by using MATLAB software. It's concluded that the saturation of water is increasing during instability phenomenon in inclined porous media when the length of fingers and time increase and without inclination, it's more increasing as compared to with inclination.

Keywords- Differential Transform Method, Inclination, Instability phenomenon, MATLAB, Non-linear partial differential equation, Saturation

I. INTRODUCTION

The study of multiphase flow through porous media has been of great significance in petroleum technology for the method of oil recovery for the last few decades. In Primary oil recovery process oil comes out from oil formatted area by natural pressure and it produces only 10% to 12% of oil by natural pressure of the reservoir. Remaining oil can recover by injecting gas, chemicals, but mostly water (reason being supply of water is plentiful, inexpensive and usually more stable frontal displacement).Since we inject fluid, this process is called the secondary process of oil recovery. In the secondary oil recovery process, the instability phenomenon occurs in the secondary oil recovery process, when water is injected into the oil-formatted field. Due to the difference in viscosity of water and oil, the protuberances will occur with irregular fingers in size and shape instead of normal displacement of the entire front (common interface).Hence, it is called fingering phenomenon. In both miscible and immiscible systems, instability may occur and arise at the interface between two fluids (e.g. oil and water). This phenomenon has been observed, studied and understanding of it has continuously evolved by many researchers. Many researchers applied different Analytical and Numerical approaches as well. Saffman and Taylor(1958)[2] suggested that fluid displacement involves only a microscopic fluid-fluid interface and it is mathematically comparable to two-dimensional flow in a porous medium. Several moving microscopic fluid-fluid interfaces are represented by the macroscopic interface and displacement are three dimensional. The first application of the Hele-Shaw cell to viscous fingering problems was made by Saffman and Taylor[2]. As both the velocity of the fluid in the porous medium and the velocity of the fluid in the Hele-Shaw cell obey Darcy's law, Saffman and Taylor proposed that the behavior of the Hele-Shaw cell could reflect the flow of the porous medium. In the displacement of one viscous liquid by another immiscible one through a uniform porous medium, Chouke et al(1959)[3] examined macroscopic instability. By defining an effective interfacial tension between the fluids in a porous medium analogous to the interfacial tension in the bulk fluids, he also accounted for capillary effects. The stability of displacement fronts in porous media was studied by Scheideggar (1960)[4]. Scheidegger and Johnson(1961)[5] examine this phenomenon analytically for the first time. They suggested statistical approach. The average cross-sectional area occupied by the fingers was considered by Scheidegger (1960)[4], while the size and shape of the individual fingers were overlooked. In a displacement phase in a

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heterogeneous porous medium with capillary pressure using a perturbation solution, Verma (1969)[6] addressed the statistical behavior of the fingering phenomenon. A non-linear evolution equation for water oil displacement front was numerically formulated by Brailovsky et al. (2006)[7]. They also explored a way of regulating the unregulated growth of fingers by not uniformly injecting water, but rather at suitably spaced time intervals. Patel and Mehta(2008)[8] have explored this phenomenon by transforming the governing equation of the fingering phenomenon into a Burger equation and an analytical solution. Joshi and Mehta (2009)[9] presented the solution of the instability phenomenon occurring in fluid flow through porous media by the Group Invariant approach. Kinjal and Mehta(2011)[10] presented a set of nonlinear partial differential equation solutions in both heterogeneous and homogeneous porous media for fingering phenomena. Kajal and Mehta(2014)[11] obtained the solution by Homotopy Analysis method. Parikh(2015)[12] obtained the solution in terms of quadratic polynomial by using generalized severable method. Borana and Prathan(2016)[13] obtained the solution by Crank-Nicolson finite difference scheme. Pathak and singh(2016)[14] obtained the solution by optimal homotopy analysis method. In the present paper, we are going to obtain solution by Differential Transform Method.

II. MATHEMATICAL FORMULATION

To formulate this phenomenon mathematically, we have considered the following assumptions. From a large natural field area, we took a vertical cross-sectional area of a piece of a cylindrical porous matrix as a rectangle having length L and inclination θ with the horizontal x-axis. It's all sides are impermeable except for one end of the cylinder which is designated as a common interface at x = 0 as shown in Fig. 1.

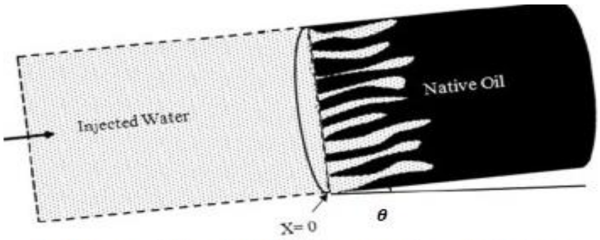

Fig. 1 Representation of instability phenomeno in a cylindrical piece of inclined hemogenous porous matrix

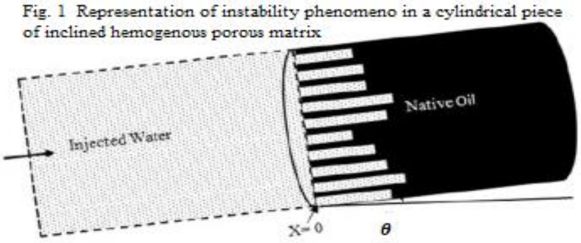

Fig. 2 Representation of Fig. 1 with rectangular fingers

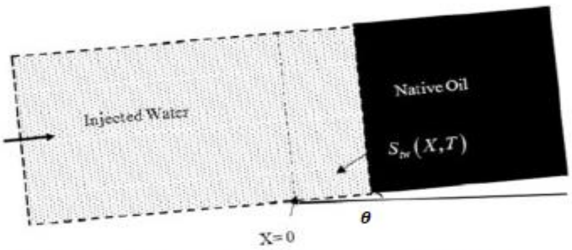

Fig. 3 Represemtation of Fig. 2 with average cross sectional area

Injected water comes into contact with oil at the common interface x = 0, the phenomenon of instability begins (Fig.1). Due to the irregular shape and size of the fingers, Scheidegger[4] suggested the schematic shape in the form of rectangular fingers (Fig. 2). However, it is still difficult to find the portion saturated with water due to an irregular length; thus, we took the average cross-section area covered by rectangular fingers (Fig. 3). The saturation of water at common interface occupied cross-sectional area is very small saturation. This study has been carried out for instability phenomenon in homogeneous porous matrix with and without inclination.

Since water is injected at common interface in inclined homogeneous porous matrix contenting oil which will displaced by injecting water. Hence water and oil both will satisfy Darcy's law given by Bear[1] which gives velocities of water and oil respectively as

$$V_{iw} = -\frac{k_{iw}}{\mu_{iw}} K \left(\frac{\partial p_{iw}}{\partial x} + \rho_{iw} g sin\theta\right) \quad (1)$$

$$V_{no} = -\frac{k_{no}}{\mu_{no}} K \left(\frac{\partial p_{no}}{\partial x} + \rho_{no} g sin\theta\right) \quad (2)$$

Since gravitational effect g and angle of inclination play an important role in the injected water and native oil velocities in the inclined porous matrix, the second terms in the above equations are applied. Here, iw represents injected water and no represents native oil. V represents

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velocity, K represents permeability of homogeneous porous medium, angle of inclination is represented by $\boldsymbol{\theta}$, k represents relative permeability, which is function of saturation S, p represents pressure. ρ represents constant density. g represents acceleration due to gravity. μ represents constant kinematic viscosity. Due to the capillary pressure of water and oil, water and oil are followed through interconnected capillaries in a porous matrix throughout the phenomenon of instability. Injected water and displaced native oil to satisfy the continuity equation given by Scheidegger as

$$\emptyset \frac{\partial S_{iw}}{\partial t} + \frac{\partial V_{iw}}{\partial x} = 0 \qquad (3)$$

$$\emptyset \frac{\partial S_{no}}{\partial t} + \frac{\partial V_{no}}{\partial x} = 0 \qquad (4)$$

Where the porosity $\emptyset$ of the homogeneous porous medium is provided as a constant here.

The sum of the saturation of the injected water and native oil is also considered to be equal to the unity (i.e. completely saturated) written as

$$S_{iw} + S_{no} = 1 \qquad (5)$$

In addition, it is important to consider the role of capillary pressure in the phenomenon of instability. Therefore, the water will flow through interconnected capillaries created by effective pores when less viscous water is injected at x = 0 in oil formatted inclined porous matrix of length x = L, and this is due to the difference in pressure of native oil and injected water. Scheidegger defined function of saturation of injected water as

$$p_c(S_{iw}) = p_{no} - p_{iw} \qquad (6)$$

However, Mehta claims that in homogeneous porous media, the injected water linearly flows through small interconnected capillaries. Therefore, he considers capillary pressure p_c as a linear S_{iw} saturation function given by

$$p_c = -\beta S_{iw}; \text{is constant.} \qquad (7)$$

The minus sign here indicates the position of the capillary pressure opposite the saturation of the water injected. This relationship between the permeability and saturation of water and oil was established by Scheidegger and Bear[1,4].

$$K_{iw} = S_{iw}\,,\ K_{no} = S_{no} = 1 - \alpha S_{iw} \qquad (8)$$

Here $\alpha = 1.11$ which is constant. The following coupled partial differential equations are now replaced by the value of the seepage velocity of injected water V_{iw} and the velocity of native oil V_{no} from eq (1) and (2) in the continuity equations of (3) and (4) as

$$\emptyset \frac{\partial S_{iw}}{\partial t} = \frac{\partial}{\partial x}\left[\frac{k_{iw}}{\mu_{iw}} K \left(\frac{\partial p_{iw}}{\partial x} + \rho_{iw} g sin\theta\right)\right] \qquad (9)$$

$$\emptyset \frac{\partial S_{no}}{\partial t} = \frac{\partial}{\partial x}\left[\frac{k_{no}}{\mu_{no}} K \left(\frac{\partial p_{no}}{\partial x} + \rho_{no} g sin\theta\right)\right] \qquad (10)$$

From equation (6) and (9), we get

$$\emptyset \frac{\partial S_{iw}}{\partial t} = \frac{\partial}{\partial x}\left[\frac{k_{iw}}{\mu_{iw}} K \left\{\left(\frac{\partial p_{no}}{\partial x}\right) - \left(\frac{\partial p_c}{\partial x}\right) + \rho_{iw} g sin\theta\right\}\right] \qquad (11)$$

Taking summation of equation (10) and (11) and from relation (5) then integrating with respect to x we get,

$$\left\{\begin{matrix} \left(\frac{k_{iw}}{\mu_{iw}} K + \frac{k_{no}}{\mu_{no}} K\right) \frac{\partial p_{no}}{\partial x} - \left(\frac{k_{iw}}{\mu_{iw}} K\right) \frac{\partial p_c}{\partial x} \\ + \left(\frac{k_{iw}}{\mu_{iw}} \rho_{iw} K + \frac{k_{no}}{\mu_{iw}} \rho_{no} K\right) gsin\theta \end{matrix}\right\} = -V(t) \qquad (12)$$

Where V(t) is a constant of integration.

From equation (12) we get $\frac{\partial p_{no}}{\partial x}$ as follows

$$\frac{\partial p_{no}}{\partial x} = \left\{\begin{matrix} -\frac{V(t)}{\left(\frac{k_{iw}}{\mu_{iw}}K + \frac{k_{no}}{\mu_{no}}K\right)} + \frac{\left(\frac{k_{iw}}{\mu_{iw}}K\right)\frac{\partial p_c}{\partial x}}{\left(\frac{k_{iw}}{\mu_{iw}}K + \frac{k_{no}}{\mu_{no}}K\right)} \\ -\frac{\left(\frac{k_{iw}}{\mu_{iw}}\rho_{iw}K + \frac{k_{no}}{\mu_{no}}\rho_{no}K\right)gsin\theta}{\left(\frac{k_{iw}}{\mu_{iw}}K + \frac{k_{no}}{\mu_{no}}K\right)} \end{matrix}\right\} \qquad (13)$$

By putting the value of $\frac{\partial p_{no}}{\partial x}$ in (13) we get

$$\emptyset \frac{\partial S_{iw}}{\partial t} = \frac{\partial}{\partial x}\left[\frac{k_{iw}}{\mu_{iw}} K \left\{\begin{matrix} -\frac{V(t)}{\left(\frac{k_{iw}}{\mu_{iw}}K + \frac{k_{no}}{\mu_{no}}K\right)} - \frac{\left(\frac{k_{no}}{\mu_{no}}K\right)\frac{\partial p_c}{\partial x}}{\left(\frac{k_{iw}}{\mu_{iw}}K + \frac{k_{no}}{\mu_{no}}K\right)} \\ -\frac{\left(\frac{k_{no}}{\mu_{no}}(\rho_{no} - \rho_{iw})K\right)gsin\theta}{\left(\frac{k_{iw}}{\mu_{iw}}K + \frac{k_{no}}{\mu_{no}}K\right)} \end{matrix}\right\}\right] \qquad (14)$$

Now Oroveanu[15] gave relation of mean pressure by

$$p_{no} = \bar{p} + \frac{1}{2} p_c, \bar{p} = \frac{(p_{no} + p_{iw})}{2} \qquad (15)$$

Where $\bar{p}$ is the constant mean pressure.

$$\frac{\partial p_{no}}{\partial x} = \frac{1}{2}\frac{\partial p_c}{\partial x} \qquad (16)$$

To determine the value of V(t) we use relation (16) in equation (13) and after simplification we get,

$$-V(t) = \left\{\begin{matrix} \frac{1}{2}\left(\frac{k_{no}}{\mu_{no}} K - \frac{k_{iw}}{\mu_{iw}} K\right)\frac{\partial p_c}{\partial x} + \\ + \left(\frac{k_{iw}}{\mu_{iw}} \rho_{iw} K + \frac{k_{no}}{\mu_{iw}} \rho_{no} K\right) gsin\theta \end{matrix}\right\} \qquad (17)$$

Hence using the value of V(t) in (14) and after simplification we get,

$$\emptyset \frac{\partial S_{iw}}{\partial t} = \frac{\partial}{\partial x}\left\{\frac{k_{iw}}{\mu_{iw}} K \left(-\frac{1}{2}\frac{\partial p_c}{\partial x}\right) + \frac{k_{iw}}{\mu_{iw}} \rho_{iw} K gsin\theta\right\} \qquad (18)$$

For more simplification of above equation of motion we use relation of p_c and k_{iw} from equation (6) and (8) respectively we get,

$$\emptyset \frac{\partial S_{iw}}{\partial t} = \frac{\beta K}{2\mu_{iw}} \frac{\partial}{\partial x}\left(S_{iw} \frac{\partial S_{iw}}{\partial x}\right) + \frac{K \rho_{iw} g}{\mu_{iw}} sin\theta \frac{\partial S_{iw}}{\partial x} \qquad (19)$$

Equation (19) is non-linear partial differential equation of motion for the saturation of injected water during instability phenomenon occurring in secondary oil recovery process in incline homogeneous porous matrix. To solve nonlinear equation (19), it is necessary to choose appropriate boundary and initial conditions at common interface x = 0.We consider here that when water is injected at common interface x = 0 the saturation of injected water is S_{w0} which is very small for given time t > 0.

i.e. $S_{iw}(0,t) = S_{w0}$ at x=0 for t > 0 (20)

$and\ S_{iw}(x,0) = S_{w0} e^{\frac{x}{L}}$ at t = 0 for x > 0 , L is length of porous matrix. (21).

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III. Differential Transform Method

Zhou[16], who solved linear and nonlinear issues in electrical circuit problems, first suggested the idea of the DTM. This technique was developed by Chen and Ho[17] for partial differential equations and Ayaz[28] applied it to the differential equation method. This approach for solving different types of equations has been used by several researchers in recent years.

IV. Two dimensional Differential Transform Method

Consider a function of two variables $w(x, t)$ and suppose that it can be represented as product of two single-variable functions, i.e., $w(x, t) = f(x)g(t)$: Based on the properties of differential transform , function $w(x,t)$ can be represented as

$$w(x,t) = \sum_{i=0}^{\infty} F(i)x^i \sum_{j=0}^{\infty} G(j)t^j = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} W(i,j)x^i t^j , \quad (22)$$

Where $W(i,j)$ = F(i)G(j) is called the spectrum of w(x,t).

If function $w(x, t)$ is analytic and differentiated continuously with respect to time t and space x in the domain of interest, then let

$$W(m\,,n) = \frac{1}{m!n!}\left[\frac{\partial^{m+n}}{\partial x^m \partial t^n} w(x,t)\right]_{\substack{x=0\\t=0}} \quad (23)$$

where the t-dimensional spectrum function $W(m,n)$ is the transformed function which is called T-function in brief.

The differential inverse transform of $W(m,n)$ is defined as follows:

$$w(x,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} W(m,n)x^m t^n \quad (24)$$

Combining (23) and (24) gives the solution as

$$w(x,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{1}{m!n!}\left[\frac{\partial^{m+n}}{\partial x^m \partial t^n} w(x,t)\right]_{\substack{x=0\\t=0}} x^m t^n \quad (25)$$

Function Form	Transformed Form
$w(x,t) = u(x,t) \pm v(x,t)$	$W(m,n) = U(m,n) \pm V(m,n)$
$w(x,t) = cu(x,t)$	$W(m,n) = cU(m,n)$
$w(x,t) = \frac{\partial}{\partial x}u(x,t)$	$W(m,n) = (m+1)U(m+1,n)$
$w(x,t) = \frac{\partial}{\partial t}u(x,t)$	$W(m,n) = (n+1)U(m,n+1)$
$w(x,t) = \frac{\partial^{r+s}}{\partial x^r \partial t^s}u(x,t)$	$W(m,n) = \frac{(m+r)!\,(n+s)!}{m!\,n!}U(m+r,n+s)$
$w(x,t) = u(x,t)v(x,t)$	$W(m,n) = \sum_{r=0}^{m}\sum_{s=0}^{n} U(r,n-s)V(m-r,s)$
$w(x,t) = x^\alpha t^\beta$	$W(m,n) = \delta(m-\alpha, n-\beta) = \begin{cases} 1 & m=\alpha, n=\beta \\ 0 & otherwise \end{cases}$

Table 1

V. Instability phenomenon in homogeneous porous media with inclination

To solve non-linear partial differential equation (19) for phenomenon of instability together with the condition (20)-(21). Choosing dimensionless variable

$X = \frac{x}{L}$ and $T = \frac{K}{\phi L^2 \mu_{iw}} t$

The equation (19) together with condition (20-21) will be converted into dimensionless form as

$$\frac{\partial S_{iw}}{\partial T} = \varepsilon\left[S_{iw}\frac{\partial^2 S_{iw}}{\partial X^2} + \left(\frac{\partial S_{iw}}{\partial X}\right)^2\right] + A\frac{\partial S_{iw}}{\partial X} \quad (26)$$

Where $\varepsilon = \frac{\beta}{2}$ and $A = \rho_{iw} g sin\theta L$ units which is constant.

$$S_{iw}(0,T) = S_{w0} \text{ at X=0 for T} > 0 \quad (27)$$

$$S_{iw}(X,0) = S_{w0}e^X \text{ at T = 0 for X} > 0 \quad (28)$$

To solve the equation (26) together with condition (27) and (28) we apply DTM method.

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Equation (26) $\frac{\partial S_{iw}}{\partial T} = \varepsilon S_{iw}\frac{\partial^2 S_{iw}}{\partial X^2} + \varepsilon\left(\frac{\partial S_{iw}}{\partial X}\right)^2 + A\frac{\partial S_{iw}}{\partial X}$ with Initial condition $S_{iw}(\mathrm{X},0) = S_{w0}e^{X}$ is transformed as follows:

$$(\mathrm{n}+1)\mathrm{S}(\mathrm{m},\mathrm{n}+1) = \varepsilon\sum_{r=0}^{m}\sum_{t=0}^{n}(\mathrm{r}+1)(\mathrm{m}-\mathrm{r}+1)\mathrm{S}(\mathrm{r}+1,\mathrm{n}-\mathrm{t})\mathrm{S}(\mathrm{m}-\mathrm{r}+1,\mathrm{t}) + \varepsilon\sum_{r=0}^{m}\sum_{t=0}^{n}(\mathrm{m}-\mathrm{r}+1)(\mathrm{m}-\mathrm{r}+2)\mathrm{S}(\mathrm{r},\mathrm{n}-\mathrm{t})\mathrm{S}(\mathrm{m}-\mathrm{r}+2,\mathrm{t}) + A(m+1)S(m+1,n)$$

(29)

Taking the values as follows:

$S_{w0} = 0.2$,$\varepsilon = 1$, $A = \rho_{iw} gLsin\theta = 0.01 \times -9.8 \times 2 \times \sin 15 = -0.05176$

By definition of DTM

$$S(m\,,n) = \frac{1}{m!n!}\left[\frac{\partial^{m+n}}{\partial X^m \partial T^n} S_{iw}(X,T)\right]_{\substack{X=0\\t=0}} \qquad (30)$$

To Apply initial condition in equation (30) putting n=T=0 we get,

$$S(m\,,0) = \frac{1}{m!0!}\left[\frac{\partial^{m}}{\partial X^m} S_{iw}(X,0)\right]_{X=0} \qquad (31)$$

Applying initial condition in equation (31) and putting different values of m ,i.e m=0,1,2,3,4,5,… we get

$$S(0\,,0) = \frac{1}{0!}\left[\frac{\partial^0}{\partial X^0} S_{iw}(X,0)\right]_{X=0}$$

$$S(0\,,0) = \frac{1}{1}\left[0.2\mathrm{e}^{\mathrm{X}}\right]_{X=0}$$

So we get S(0,0) =0.2

$$S(1\,,0) = \frac{1}{1!}\left[\frac{\partial^1}{\partial X^1} S_{iw}(X,0)\right]_{X=0}$$

$$S(1\,,0) = \frac{1}{1}\left[\frac{\partial^1}{\partial X^1} 0.2\mathrm{e}^{\mathrm{X}}\right]_{X=0}$$

$$S(1\,,0) = \left[0.2\mathrm{e}^{\mathrm{X}}\right]_{X=0}$$

So we get S(1,0) = 0.2

$$S(2\,,0) = \frac{1}{2!}\left[\frac{\partial^2}{\partial X^2} S_{iw}(X,0)\right]_{X=0}$$

$$S(2\,,0) = \frac{1}{2}\left[\frac{\partial^2}{\partial X^2} 0.2\mathrm{e}^{\mathrm{X}}\right]_{X=0}$$

$$S(2\,,0) = \frac{1}{2}\left[0.2\mathrm{e}^{\mathrm{X}}\right]_{X=0}$$

So we get S(2,0) = $\frac{0.2}{2}$

In similar manner, we will get S(3,0) = $\frac{0.2}{6}$

S(4,0) = $\frac{0.2}{24}$

S(5,0) = $\frac{0.2}{120}$

Therefore, the values that we found are

S(0,0) = 0.2 , S(1,0) = 0.2 , S(2,0) = $\frac{0.2}{2}$, S(4,0) = $\frac{0.2}{24}$, S(5,0) = $\frac{0.2}{120}$

Putting different values of m and n in equation (29) we got different S(X,T) which are as follows:

S(0,0) = 0.2	S(1,0) = 0.2	S(2,0) = $\frac{0.2}{2}$	S(3,0) = $\frac{0.2}{6}$
S(0,1) = 0.0696	S(1,1) = 0.1392	S(2,1) = 0.1392	S(3,1) = 0.0928
S(0,2) = 0.00643	S(1,2) = 0.01929	S(2,2) =0.0145	S(3,2) = 0.0289
S(0,3) = 0.000567	S(1,3) = 0.002268	S(2,3) = 0.004536	S(3,3) = 0.0060

And so on…

Now by equation (24) we get,

$$S_{iw}(X,T) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} S(m,n)X^m T^n$$

so, $S_{iw}(X,T) = S(0,0)X^0T^0 + S(1,0)X^1T^0 + S(2,0)X^2T^0 + S(3,0)X^3T^0 + S(0,1)X^0T^1 + S(1,1)X^1T^1 + S(2,1)X^2T^1 + S(3,1)X^3T^1 + S(0,2)X^0T^2 + S(1,2)X^1T^2 + S(2,2)X^2T^2 + S(3,2)X^3T^2 + S(0,3)X^0T^3 + S(1,3)X^1T^3 + S(2,3)X^2T^3 + S(3,3)X^3T^3 + \cdots$

Putting the value from the above table, we get,

$$S_{iw}(X,T) = 0.2 + 0.2X + \frac{0.2X^2}{2!} + \frac{0.2X^3}{3!} + 0.0696T + 0.1392XT + 0.1392X^2T + 0.0928X^3T + 0.00643T^2 + 0.01929XT^2 + 0.0145X^2T^2 + 0.0289X^3T^2 + 0.000567T^3 + 0.002268XT^3 + 0.004536X^2T^3 + 0.0060X^3T^3 + \cdots \qquad (32)$$

The above equation shows the saturation of injected water.

VI. Numerical and Graphical Presentation with Inclination

Numerical and Graphical presentations of Equation (32) has been obtained using MATLAB coding. Figure 2 shows the graph of saturation of injected water $S_{iw}(X,T)$ vs. X for fixed time T=0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9,1.0 , The following table represents the numerical values for height for different distance x for fixed time t=0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9,1.0 .

	Saturation of the injected water in inclined porous matrix									
X/T	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.1	0.2288	0.2367	0.2448	0.2537	0.2616	0.2702	0.2789	0.2880	0.2972	0.3066
0.2	0.2528	0.2617	0.2707	0.2806	0.2893	0.2989	0.3088	0.3189	0.3293	0.3399

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0.3	0.2794	0.2892	0.2992	0.3103	0.3201	0.3309	0.3419	0.3533	0.3650	0.3770
0.4	0.3088	0.3197	0.3309	0.3431	0.3541	0.3662	0.3787	0.3916	0.4048	0.4184
0.5	0.3414	0.3534	0.3658	0.3795	0.3918	0.4054	0.4195	0.4340	0.4490	0.4645
0.6	0.3773	0.3907	0.4045	0.4198	0.4336	0.4489	0.4648	0.4813	0.4984	0.5160
0.7	0.4170	0.4318	0.4473	0.4644	0.4799	0.4972	0.5152	0.5340	0.5534	0.5736
0.8	0.4609	0.4774	0.4946	0.5138	0.5313	0.5509	0.5713	0.5926	0.6149	0.6381
0.9	0.5094	0.5278	0.5470	0.5685	0.5883	0.6105	0.6337	0.6581	0.6836	0.7104
1.0	0.5630	0.5835	0.6050	0.6292	0.6516	0.6767	0.7033	0.7312	0.7607	0.7917

Table 2. Numerical values of saturation of injected water in Homogeneous Porous Matrix with small inclination $\theta = 15^{\circ}$

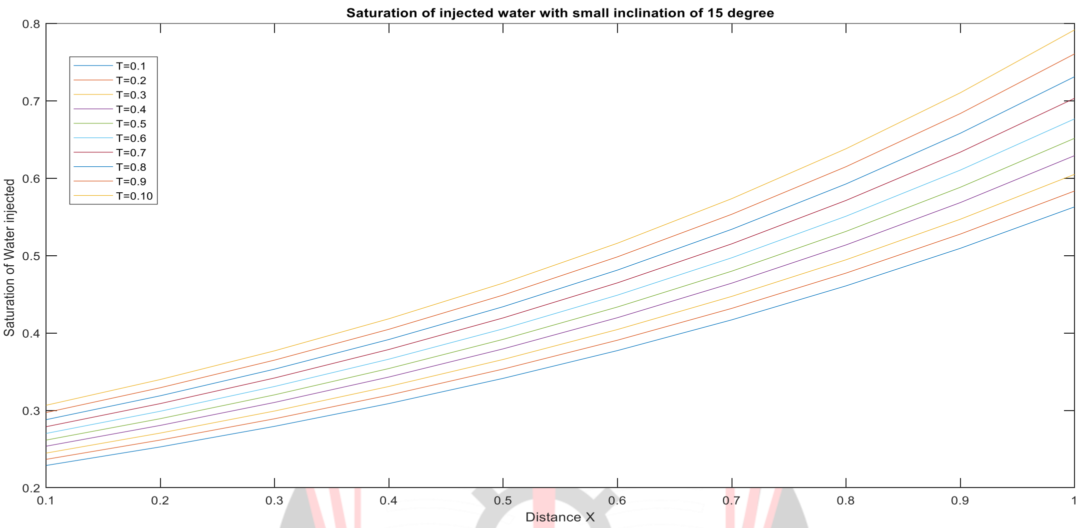

Fig. 4

VII DEDUCTION OF INSTABILITY PHENOMENON

in homogeneous porous media without inclination:

To solve non-linear partial differential equation (19) for phenomenon of instability together with the condition (20)-(21). Choosing dimensionless variable

$X = \frac{x}{L}$ and $T = \frac{K}{\emptyset L^2 \mu_{iw}} t$

The equation (19) together with condition (20-21) will be converted into dimensionless form as

$$\frac{\partial S_{iw}}{\partial T} = \varepsilon \left[S_{iw} \frac{\partial^2 S_{iw}}{\partial X^2} + \left(\frac{\partial S_{iw}}{\partial X} \right)^2 \right] + A \frac{\partial S_{iw}}{\partial X} \quad (33)$$

Where $\varepsilon = \frac{\beta}{2}$ and $A = \rho_{iw} g sin\theta L$ units which is constant.

$$S_{iw}(0, T) = S_{w0} \text{ at X=0 for T} > 0 \quad (34)$$

$$S_{iw}(X, 0) = S_{w0} e^X \text{ at T = 0 for X} > 0 \quad (35)$$

To solve the equation (33) together with condition (34) and (35) we apply DTM method.

Equation (33) $\frac{\partial S_{iw}}{\partial T} = \varepsilon S_{iw} \frac{\partial^2 S_{iw}}{\partial X^2} + \varepsilon \left(\frac{\partial S_{iw}}{\partial X} \right)^2 + A \frac{\partial S_{iw}}{\partial X}$ with Initial condition $S_{iw}(\mathrm{X}, 0) = S_{w0} e^X$ is transformed as follows:

$$\begin{aligned}(\mathrm{n}+1)\mathrm{S}(\mathrm{m}, \mathrm{n}+1) &= \varepsilon \sum_{\mathrm{r}=0}^{\mathrm{m}} \sum_{\mathrm{t}=0}^{\mathrm{n}} (\mathrm{r}+1)(\mathrm{m}-\mathrm{r}+1)\mathrm{S}(\mathrm{r}+1, \mathrm{n}-\mathrm{t})\mathrm{S}(\mathrm{m}-\mathrm{r}+1, \mathrm{t}) \\ &+ \varepsilon \sum_{\mathrm{r}=0}^{\mathrm{m}} \sum_{\mathrm{t}=0}^{\mathrm{n}} (\mathrm{m}-\mathrm{r}+1)(\mathrm{m}-\mathrm{r}+2)\mathrm{S}(\mathrm{r}, \mathrm{n}-\mathrm{t})\mathrm{S}(\mathrm{m}-\mathrm{r}+2, \mathrm{t}) + A(m+1)S(m+1, n)\end{aligned} \quad (36)$$

Taking the values as follows:

$S_{w0} = 0.2$, $\varepsilon = 1$, $A = \rho_{iw} g L sin\theta = 0.01 \times -9.8 \times 2 \times \sin 0 = 0$

By definition of DTM

$$S(m, n) = \frac{1}{m!n!} \left[\frac{\partial^{m+n}}{\partial X^m \partial T^n} S_{iw}(X, T) \right]_{\substack{X=0 \\ t=0}} \quad (37)$$

To Apply initial condition in equation (30) putting n=T=0 we get,

$$S(m, 0) = \frac{1}{m!0!} \left[\frac{\partial^m}{\partial X^m} S_{iw}(X, 0) \right]_{X=0} \quad (38)$$

Applying initial condition in equation (36) and putting different values of m , i.e m=0,1,2,3,4,5,… we get

$$S(0, 0) = \frac{1}{0!} \left[\frac{\partial^0}{\partial X^0} S_{iw}(X, 0) \right]_{X=0}$$

$$S(0, 0) = \frac{1}{1} [0.2\mathrm{e}^{\mathrm{X}}]_{X=0}$$

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So we get S(0,0) =0.2

$S(1\,,0)\,=\frac{1}{1!}\left[\frac{\partial^1}{\partial X^1}S_{iw}(X,0)\right]_{X=0}$

$S(1\,,0)\,=\frac{1}{1}\left[\frac{\partial^1}{\partial X^1}0.2\mathrm{e}^{\mathrm{X}}\right]_{X=0}$

$S(1\,,0)\,=[0.2\mathrm{e}^{\mathrm{X}}]_{X=0}$

So we get S(1,0) = 0.2

$S(2\,,0)\,=\frac{1}{2!}\left[\frac{\partial^2}{\partial X^2}S_{iw}(X,0)\right]_{X=0}$

$S(2\,,0)\,=\frac{1}{2}\left[\frac{\partial^2}{\partial X^2}0.2\mathrm{e}^{\mathrm{X}}\right]_{X=0}$

$S(2\,,0)\,=\frac{1}{2}[0.2\mathrm{e}^{\mathrm{X}}]_{X=0}$

So we get S(2,0) = $\frac{0.2}{2}$

In similar manner, we will get S(3,0) = $\frac{0.2}{6}$

S(4,0) = $\frac{0.2}{24}$

S(5,0) = $\frac{0.2}{120}$

Therefore, the values that we found are

S(0,0) = 0.2 , S(1,0) = 0.2 , S(2,0) = $\frac{0.2}{2}$, S(4,0) = $\frac{0.2}{24}$, S(5,0) = $\frac{0.2}{120}$

Putting different values of m and n in equation (29) we got different S(X,T) which are as follows:

S(0,0) = 0.2	S(1,0) = 0.2	S(2,0) = $\frac{0.2}{2}$	S(3,0) = $\frac{0.2}{6}$
S(0,1) = 0.08	S(1,1) = 0.16	S(2,1) = 0.16	S(3,1) = $\frac{0.32}{3}$
S(0,2) = 0.0072	S(1,2) = 0.0216	S(2,2) = 0.0162	S(3,2) = 0.0324
S(0,3) = 0.0006	S(1,3) = 0.0024	S(2,3) = 0.0048	S(3,3) = 0.0064

And so on…

Now by equation (24) we get,

$S_{iw}(X,T) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} S(m,n)X^mT^n$

so, $S_{iw}(X,T) = S(0,0)X^0T^0 + S(1,0)X^1T^0 + S(2,0)X^2T^0 + S(3,0)X^3T^0 + S(0,1)X^0T^1 + S(1,1)X^1T^1 + S(2,1)X^2T^1 + S(3,1)X^3T^1 + S(0,2)X^0T^2 + S(1,2)X^1T^2 + S(2,2)X^2T^2 + S(3,2)X^3T^2 + S(0,3)X^0T^3 + S(1,3)X^1T^3 + S(2,3)X^2T^3 + S(3,3)X^3T^3 + \cdots$

Putting the value from the above table, we get,

$S_{iw}(X,T) = 0.2 + 0.2X + \frac{0.2X^2}{2!} + \frac{0.2X^3}{3!} + 0.08T + 0.16XT + 0.16X^2T + \frac{0.32}{3}X^3T + 0.0072T^2 + 0.0216XT^2 + 0.0162X^2T^2 + 0.0324X^3T^2 + 0.0006T^3 + 0.0024XT^3 + 0.0048X^2T^3 + 0.0064X^3T^3 + \cdots$ (39)

The above equation shows the saturation of injected water.

VIII Numerical and Graphical presentation without inclination

Numerical and Graphical presentations of Equation (39) has been obtained using MATLAB coding. Figure 2 shows the graph of saturation of injected water $S_{iw}(X,T)$ vs. X for fixed time T=0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9,1.0 , The following table represents the numerical values for height for different distance x for fixed time t= 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0 .

	S_{iw} (X,T)									
X/T	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	0.2299	0.2391	0.2484	0.2579	0.2675	0.2774	0.2875	0.2978	0.3083	0.3191
0.2	0.2542	0.2642	0.2746	0.2852	0.2960	0.3070	0.3183	0.3299	0.3417	0.3538
0.3	0.2798	0.2921	0.3036	0.3154	0.3274	0.3398	0.3525	0.3655	0.3789	0.3926
0.4	0.3104	0.3229	0.3357	0.3488	0.3623	0.3762	0.3904	0.4051	0.4202	0.4357
0.5	0.3431	0.3569	0.3711	0.3858	0.4009	0.4165	0.4326	0.4492	0.4663	0.4839
0.6	0.3792	0.3946	0.4104	0.4268	0.4437	0.4613	0.4794	0.4982	0.5176	0.5377
0.7	0.4191	0.4362	0.4538	0.4722	0.4912	0.5110	0.5315	0.5528	0.5750	0.5979
0.8	0.4633	0.4822	0.5019	0.5225	0.5438	0.5662	0.5895	0.6138	0.6397	0.6654
0.9	0.5120	0.5331	0.5551	0.5782	0.6023	0.6276	0.6541	0.6818	0.7108	0.7412
1.0	0.5639	0.5894	0.6140	0.6399	0.6672	0.6959	0.7261	0.7578	0.7913	0.8237

Table 3. Numerical values of saturation of injected water in Homogeneous Porous Matrix without inclination $\theta = 15^{\circ}$

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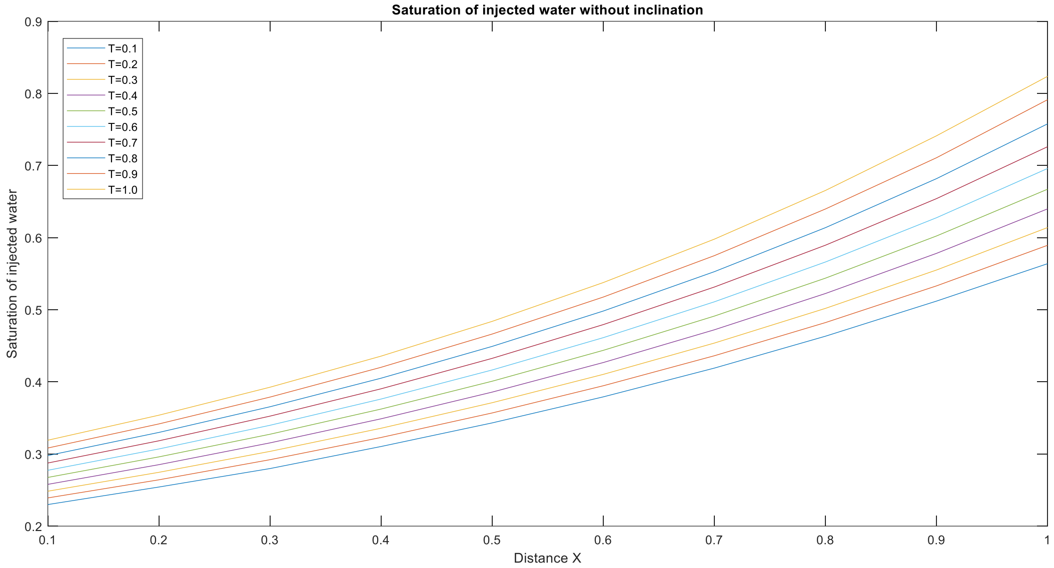

Fig. 5

	Saturation of the injected water in inclined porous matrix									
	T=0.1		T=0.2		T=0.3		T=0.4		T=0.5	
X/T	With incline	Without incline	With incline	Without incline	With incline	Without incline	With incline	Without incline	With incline	Without incline
0.1	0.2288	0.2299	0.2367	0.2391	0.2448	0.2484	0.2537	0.2579	0.2616	0.2675
0.2	0.2528	0.2542	0.2617	0.2642	0.2707	0.2746	0.2806	0.2852	0.2893	0.2960
0.3	0.2794	0.2798	0.2892	0.2921	0.2992	0.3036	0.3103	0.3154	0.3201	0.3274
0.4	0.3088	0.3104	0.3197	0.3229	0.3309	0.3357	0.3431	0.3488	0.3541	0.3623
0.5	0.3414	0.3431	0.3534	0.3569	0.3658	0.3711	0.3795	0.3858	0.3918	0.4009
0.6	0.3773	0.3792	0.3907	0.3946	0.4045	0.4104	0.4198	0.4268	0.4336	0.4437
0.7	0.4170	0.4191	0.4318	0.4362	0.4473	0.4538	0.4644	0.4722	0.4799	0.4912
0.8	0.4609	0.4633	0.4774	0.4822	0.4946	0.5019	0.5138	0.5225	0.5313	0.5438
0.9	0.5094	0.5120	0.5278	0.5331	0.5470	0.5551	0.5685	0.5782	0.5883	0.6023
1.0	0.5630	0.5639	0.5835	0.5894	0.6050	0.6140	0.6292	0.6399	0.6516	0.6672

	Saturation of the injected water in inclined porous matrix									
	T=0.6		T=0.7		T=0.8		T=0.9		T=1.0	
X/T	With incline	Without incline	With incline	Without incline	With incline	Without incline	With incline	Without incline	With incline	Without incline
0.1	0.2702	0.2774	0.2789	0.2875	0.2880	0.2978	0.2972	0.3083	0.3066	0.3191
0.2	0.2989	0.3070	0.3088	0.3183	0.3189	0.3299	0.3293	0.3417	0.3399	0.3538
0.3	0.3309	0.3398	0.3419	0.3525	0.3533	0.3655	0.3650	0.3789	0.3770	0.3926
0.4	0.3662	0.3762	0.3787	0.3904	0.3916	0.4051	0.4048	0.4202	0.4184	0.4357
0.5	0.4054	0.4165	0.4195	0.4326	0.4340	0.4492	0.4490	0.4663	0.4645	0.4839
0.6	0.4489	0.4613	0.4648	0.4794	0.4813	0.4982	0.4984	0.5176	0.5160	0.5377
0.7	0.4972	0.5110	0.5152	0.5315	0.5340	0.5528	0.5534	0.5750	0.5736	0.5979

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0.8	0.5509	0.5662	0.5713	0.5895	0.5926	0.6138	0.6149	0.6397	0.6381	0.6654
0.9	0.6105	0.6276	0.6337	0.6541	0.6581	0.6818	0.6836	0.7108	0.7104	0.7412
1.0	0.6767	0.6959	0.7033	0.7261	0.7312	0.7578	0.7607	0.7913	0.7917	0.8237

Table 4. Comparative Numerical Value of saturation of injected water with and without inclination.

IX. Conclusion

Equations (31) and (39) shows the saturation of injected water. Equation (31) represents saturation of injected water with inclination of small degree and Equation (39) represents saturation of injected water without inclination. Table 2 and Fig. 4 gives numerical and graphical solutions of equation (31). Table 3 and Fig. 5 gives numerical and graphical solutions of equation (39).

We can conclude that Saturation of injected water increases exponentially as distance X increases. Other thing that we observe is, if we look at table 4, we can see that due to inclination the saturation of injected water is slightly less comparative to without inclination. Saturation shows how much instability is in there and how can we adjust the angle to get the suitable result. Many researchers have applied many methods for this instability phenomenon. Compared to those methods Differential transform method gets result faster.

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