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On characterization Integral inequalities for C-totally real submanifolds in Sasakian space forms

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Abstract - We give two intrinsic integral inequalities for compact minimal C-totally real sub manifolds in a Sasakian space form.

Key words: C-totally real submanifold, Sasakian space forms , Riemannian manifold

I. §1. Introduction

Let $\bar{M}^{2m+1}$ be an odd dimensional Riemannian manifold with metric g. Let Φ be a (1, 1)-tensor field, η a 1-form on $\bar{M}^{2m+1}$ and ζ a vector field, such that

$$\begin{cases} \varphi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1 \\ g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X,\xi) \end{cases}$$

If, in addition, d η (X, Y) = g(φX, Y) for all vector fields X, Y on $\bar{M}^{2m+1}$, then $\bar{M}^{2m+1}$ is said to have a contact metric structure (φ, ξ, η, g), and $\bar{M}^{2m+1}$ is called a contact metric manifold. If, moreover, the structure is normal, that is if

$$[\phi X, \phi Y] + \phi^2[X,Y] - \phi[X,\phi Y] - \phi[\phi X, Y] = -2d\eta\,(X,Y)\xi$$

then the contact metric structure is a Sasakian structure (normal contact metric structure) and M¯ 2m+1 is called a Sasakian manifold. For details and background, see the standard references [4] and [5].

A plane section σ in $TP\bar{M}^{2m+1}$ of a Sasakian manifold $\bar{M}^{2m+1}$ is called a φ-section if it is spanned by X and φX, where X is a unit tangent vector field orthogonal to ξ. The sectional curvature $\bar{K}$ (σ) with respect to a φ-section σ is called a φ-section curvature. If a Sasakian manifold $\bar{M}^{2m+1}$ has a constant φ-sectional curvature c, then $\bar{M}^{2m+1}$ is called a Sasakian space form and is denoted by $\bar{M}^{2m+1}(c)$.

An n-dimensional submanifold M^n of a Sasakian space form $\bar{M}^{2m+1}(c)$ is called a C-totally real submanifold of $\bar{M}^{2m+1}(c)$, if ξ is a normal vector field on M^n. A direct consequence of this definition is that $\varphi(TM^n) \subset T^{\perp}M^n$, which means that M^n an anti-invariant submanifold of $\bar{M}^{2m+1}(c)$.

In [1,2], Cao gave an integral inequality for compact pseudo-umbilical space-like submanifolds in the indefinite space form. In this paper, we prove Cao's result in the case of submanifolds in the Sasakian space. We will prove the following.

Theorem1. Let M^n be an n-dimensional compact C-totally real submanifold in the Sasakain space form $M^{2n+1}(c)$; then

$$\int_{M^n} \left\{ \frac{1}{2}\sum R^2_{mijk} - \sum R^2_{mj} + \frac{1}{8}[2n(n-1)](c+3)\rho \right\} * 1 \le 0$$

Theorem 2. Let Mn be an n-dimensional compact C-totally real submanifold in the Sasakian space form M¯ 2n+1(c); then

$$\int_{M^n} \left\{ \frac{1}{2}\sum R^2_{mijk} - \sum R^2_{mj} - \left[\frac{2n(n-1)-1}{8}\right](c+3)|h|^2 \right\} * 1$$
$$\le \frac{-2n^2(n-1) + n(n-1)}{32}(c+3).\, vol\,(M^n)$$

In the above theorems, $\sum R^2_{mijk}$ is the square length of Riemannian curvature tensor of M^n and ρ is the scalar curvature of M^n

II. §2. Local formulae

We shall give the structure equations of an n-dimensional submanifold M^n of a Sasakian Space form $\bar{M}^{2m+1}(c)$. We choose a local field of orthonormal frames.

$$\begin{cases} e_1, e_2, \ldots\ldots, e_n, e_{n+1}, \ldots\ldots\ldots, e_m: \qquad e_{0*} = \xi, \\ e_{1*=\phi}e_1, \ldots\ldots\ldots, e_{n*} = \phi e_n: \ e_{(n+1)^*=\phi e_{n+1}} \ldots\ldots\ldots\ldots, e_{m*} = \varphi e_m \end{cases}$$

on $\bar{M}^{2m+1}(c)$ in such a way that, restricted to M^n, the vectors $e_1, e_2, \ldots, e_n$ are tangent to M^n, and hence $e_{n+1}, \ldots, e_m$, ξ, $e_{1*}, e_{2*}, \ldots, e_{m*}$, . . .are normal to M^n. Let $w_1, w_2, \ldots, w_n, w_{n+1}, \ldots, w_m, w_{(n+1)*}, \ldots, w_m, w_{n+1} \ldots, w_{2*}, \ldots, w_{m*}, w_{(n+1)*}, \ldots, w_{m*}$ be the field of dual frames with respect to this frame field of $\bar{M}^{2m+1}(c)$. We shall make use of the following convention on the ranges of indices:

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$$\begin{cases} A, B, C, \dots = 1, \dots, m, 0^*, 1^*, \dots, m^* \\ i, j, k, \dots\dots = 1,2 \dots \dots, n, \\ a, b, c, \dots = (n+1), \dots, m, 0^*, 1^*, \dots\dots, m^* \end{cases}$$

Then the structure equation of $\bar{M}^{2n+1}$are given by

$$\begin{cases} dw_A = -\sum \omega_{AB} \wedge \omega_{B,} \omega_{AB} + \omega_{BA,} = 0 \\ dw_{AB} = -\sum \omega_{AC} \wedge \omega_{CB,} + \frac{1}{2} \sum R_{ABCD}\, \omega_C \wedge \omega_{D,} \end{cases}$$

We restrict these forms to Mn. Then $w_a = 0$. Since $0 = dw_a = -\sum \omega_a = 0$. Since $0 = d\omega_a = -\sum \omega_{ai} \wedge \omega_{ii}$, by Cartan's Lemma, we obtain

$$\omega_{ai} = \sum h_{ij}^a\, \omega_j, \quad h_{ij}^a = h_{ji}^a.$$

Integral inequalities for C-totally real submanifolds

From these formulas, we obtain the structure equations of M^n

(2.1) $$\begin{cases} dw_i = -\sum \omega_{ik} \wedge \omega_{k,}, \omega_{ik} + \omega_{ki,} = 0 \\ dw_{AB} = -\sum \omega_{ik} \wedge \omega_{kj,} + \frac{1}{2} \sum R_{ijkl} \wedge \omega_t \end{cases}$$

(2.2) $$R_{ijkl} = \frac{1}{4}(c+3)\left((\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum (h_{ik}^a h_{jl}^a - h_{il}^a h_{jk}^a\right)$$

Where R_{ijkl} are the components of the curvature tensor of M^n. We call

$$h = \sum h_{ij}^a \omega_i \otimes \omega_j e_a$$

the second fundamental form of Mn. The square length of h is

$$|h|^2 = \sum (h_{ij}^a)^2$$

and the mean curvature of $M^n is\ H = \frac{1}{n} \sum h_{ii}^a e_a$. If M^n is minimal, the

(2.3) $$\sum h_{ii}^a = 0.$$

Let h_{ijk}^a and h_{ijkl}^a denote the covariant derivative and second covariant derivative of h_{ij}^a respectively, defined by

$$\sum h_{ijk}^a w_k = dh_{ij}^a - \sum h_{ik}^a w_{kj} - \sum h_{jk}^a w_{ki}$$
$$\sum h_{ijkl}^a w_l = dh_{ijk}^a - \sum h_{ijl}^a w_{lk} - \sum h_{ilk}^a w_{lj} - \sum h_{ljk}^a w_{li}.$$

Then we have

(2.4) $$\sum h_i^a - \sum h_{ik}^a = 0$$

(2.5) $$\sum h_i^a - \sum h_{ik}^a = \sum h_{im}^a R_{mjkl} - \sum h_{hm}^a R_{mikl}$$

The Laplacian $\triangle h_{ij}^a$ of h_{ij}^a is defined as $\sum h_{ijkk}^a$ and from Lemma 3.3 in [4], (2.3), (2.4) and (2.5), we have (as in [3, 1])

(2.6) $$\triangle h_{ij}^a = \sum h_{im}^a R_{mkjk} - \sum h_{hm}^a R_{mijk}$$

Proof of Theorem 1. From (2.2), (2.3) and (2.6),

$$\begin{aligned} \sum h_i^a \triangle h_{ij}^a &= \sum h_{ij}^a h_{mk}^a R_{mijk} + \sum h_{ij}^a h_{hm}^a R_{mkjk} \\ &= \frac{1}{2} \sum (h_{ij}^a h_{mk}^a - h_{mj}^a h_{ik}^a)\, R_{mikl} + \sum (h_{ij}^a h_{im}^a - h_{ii}^a h_{im}^a)\, R_{mj} \\ &= \frac{1}{2} \sum [\tfrac{1}{4}(c+3)\ (\delta_{ij}\delta_{mk} - \delta_{mj}\delta_{ik})] - R_{imjk}]\ R_{imjk} \\ &\quad + \sum [\tfrac{1}{4} n(n-1)(c+3)\ \delta_{ij} - R_{mj}]\ R_{mj} \\ &= \frac{1}{2} \sum R_{mijk}^2 - \sum R_{mj}^2 + \frac{1}{8}[2n(n-1) - 1](c+3)\rho. \end{aligned}$$

$$\int_{M^n} \left\{ \sum h_{ij}^a \triangle h_{ij}^a \right\} * 1 \le 0,$$

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We get

$$\int_{M^n} \left\{\frac{1}{2}\sum R_{mijk}^2 - \sum R_{mj}^2 + \frac{1}{8}[2n(n-1)-1](c+3)\rho\right\}$$

and Theorem 1 is proved,

Proof of Theorem 2. From (2.2) and (2.3), we infer

(2.8) $\rho = \frac{1}{4}n(n-1)(c+3) - |h|^2$.

From (2.7) and (2.8), we get

$$\int_{M^n} \left\{\frac{1}{2}\sum R_{mijk}^2 - \sum R_{mj}^2 + \left(\frac{2n(n-1)-1}{8}\right)(c+3)|h|^2\right\} * 1$$
$$\leq \frac{-2n^2(n-1)^2 + n(n-1)}{32}(c+3).vol(M^n),$$

Which concludes the proof.

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