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# Development of an Estimation Algorithm

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**Abstract Estimation algorithms are the building block for many modern control systems strategies. Control system Observers/Identifiers designs for example are based on these algorithms. In this paper a new algorithm is proposed. Its derivation will be based on linear algebra's vector and matrix analysis. Theorems and proofs are presented that lead to the final form**

**Keywords Estimation equation, Estimation algorithms, Control system pa- rameters, Control system states, Kalman Filter, and Recursive Least-Squares.**

## I. Introduction

Consider the difference equation,

$$\hat{a}(k+1) = \alpha(k)\,\hat{a}(k) + \beta(k)a \qquad (1)$$

where $\hat{a}$ is an $n \times 1$ estimate vector, $a$ is an $n \times 1$ constant input vector, and $\alpha(k)$ and $\beta(k)$ are time-variant $n \times n$ coefficient matrices. The question to be answered is, what set of conditions can be imposed on $\alpha(k)$ and $\beta(k)$ so that the estimate vector $\hat{a}$ will approach $a$ in steady state. In the following section, two theorems will be discussed that will offer an answer to this question, see [1].

## II. Formulation of Estimation Equation

**Theorem 1.** *Assume that the matrices $\alpha(k)$ and $\beta(k)$ in Equation (1) satisfy the following conditions.*

(1) *The time response of the homogeneous equation*
$\hat{a}(k+1) = \alpha(k)\,\hat{a}(k)$
*associated with (1), approaches zero for any $\hat{a}_0$ as $k \to \infty$.*

(2) $\alpha(k) = I - \beta(k)$.

*Then $\hat{a}(k) \to a$, as $k \to \infty$ for any constant a.*

***Proof.*** From Eq. (1) with $\alpha(k) = I - \beta(k)$, there results

$$\hat{a}(k+1) = (I - \beta(k))\,\hat{a}(k) + \beta(k)a. \qquad (2)$$

Let $\hat{d}(k) = \hat{a} - a$. From Eq. (2) it follows that

$$\hat{d}(k+1) = (I - \beta(k))\hat{d}(k)$$

$$\hat{d}(k+1) = \alpha(k)\hat{d}(k).$$

But from condition (1), $\hat{d}(k) \to 0$ *as* $k \to \infty$ and so $\hat{a}(k)$ must approach a. □

Next condition (1) of Theorem 1 is investigated. The following theorem gives sufficient but not a necessary requirement for condition (1) in Theorem 1 to be true.

**Theorem 2.** *If there exist an integer $j \geq 1$ and constant $\sigma < 1$ such that for any given norm*

$$||I - \beta(k)|| < \sigma$$

*for all $k \geq j$, then condition (1) of Theorem 1 is satisfied.*

*Proof.* In view of Theorem 1, the term of the unforced response vector $\hat{a}$ at the iteration *k+1* is

$$||\,\hat{a}(k+1)|| = ||\,I - \beta(k)\hat{a}(k)|| \leq ||I - \beta(k)||\,||\hat{a}(k)||.$$

By assumption for $k \geq j$,

$$||\,\hat{a}(k+1)|| = \sigma^k ||\hat{a}(k)|| \to 0 \text{ as } k \to \infty.$$

Since $\sigma < 1$. □

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As a demonstration, consider the following two examples:

**Example 1**.

Let

$\beta(k) = \begin{bmatrix} k/(1+k) & 1/k \\ 1/(1+k) & 0.8 \end{bmatrix}$, and $a^T = [3 \quad 1]$,

Then from Eq. (2)

$$\hat{a}(k+1) = \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} k/(1+k) & 1/k \\ 1/(1+k) & 0.8 \end{bmatrix} \right) \hat{a}(k) + \begin{bmatrix} k/(1+k) & 1/k \\ 1/(1+k) & 0.8 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

In this example, for $\sigma = 0.7$ and $j = 2$, the Euclidean norm of the matrix $(I - \beta(k))$ satisfies Theorem 2, see Fig. 1. As a consequence, the norm of the vector $\hat{a}(k)$ for the time response of the homogeneous equation associated with Eq. 2 converges to zero as can be seen from Fig. 2.

$$\hat{a}(k+1) = (I - \beta(k))\, \hat{a}(k) \qquad (3)$$

$$\hat{a}(k+1) = \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} k/(1+k) & 1/k \\ 1/(1+k) & 0.8 \end{bmatrix} \right) \hat{a}(k).$$

For an initial value of $a^T = [2.2 \quad 1.3]$, the vector $\hat{a}(k)$ converges to vector **a**, as $k$ grows large, see Fig. 3.

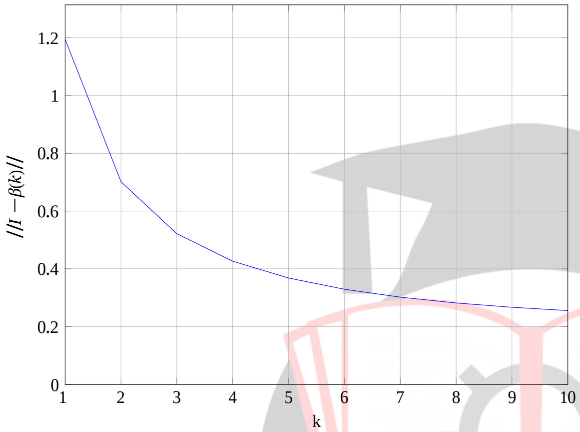


Figure 1 : Norm of the matrix $(I - \beta(k))$

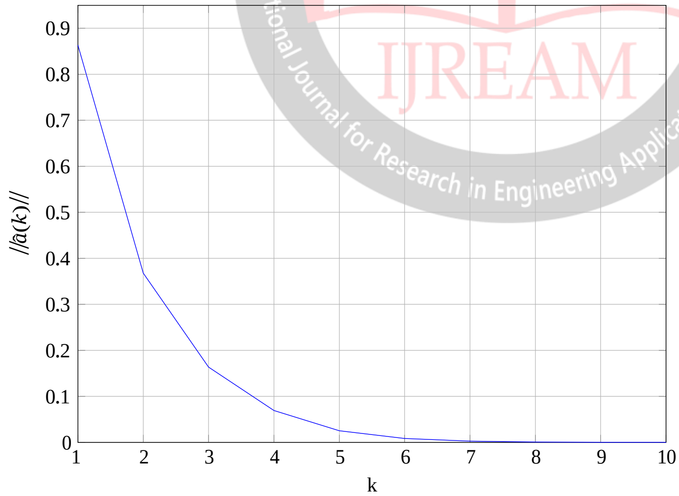


Figure 2 : Norm of the vector $\hat{a}(k)$

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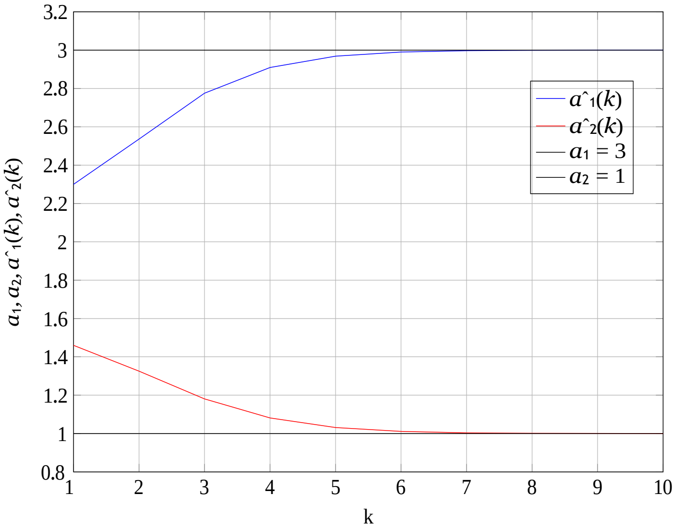


Figure 3 : Vector **a** and its estimate $\hat{a}(k)$

**Example 2.**

Consider the same previous example except for the input vector $a$ is not constant, i.e.,

$a^T(k) = [1 + 0.05k, \quad \sin(2\pi k/75)]$

Again, according to Eq. (2),

$$\hat{a}(k+1) = \left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \begin{bmatrix}k/(1+k) & 1/k\\1/(1+k) & 0.8\end{bmatrix}\right)\hat{a}(k) + \begin{bmatrix}k/(1+k) & 1/k\\1/(1+k) & 0.8\end{bmatrix}\begin{bmatrix}1+0.05k\\\sin(2\pi k/75)\end{bmatrix}$$

Fig. 4 shows a plot of the input vector $a(k)$ and its estimate $\hat{a}(k)$. In this example, only an approximate estimate is obtained because the input vector $a(k)$ is not constant. It is observed that the faster the decay of the time response of the homogeneous equation in Eq. 2, the better the estimate becomes. To show this, the matrix $\beta(k)$ of Example 1 is modified to

$$\beta(k) = \begin{bmatrix}k/(1+k) & 1/k\\1/(1+k) & 0.3\end{bmatrix},$$

so that it will cause the system of Eq. (3) to have a slower decaying unforced response. A comparison of Fig. 2 and Fig. 5 will show this. Fig. 6 shows the result of this modification on the estimated input vector$\hat{a}(k)$. It can be concluded then that the rate of convergence to zero of the norm of $\hat{a}(k)$ in Eq. (3) has direct relation to its tracking of vector **a** in Eq. (2). The faster the decay, the more accurate is the estimate.

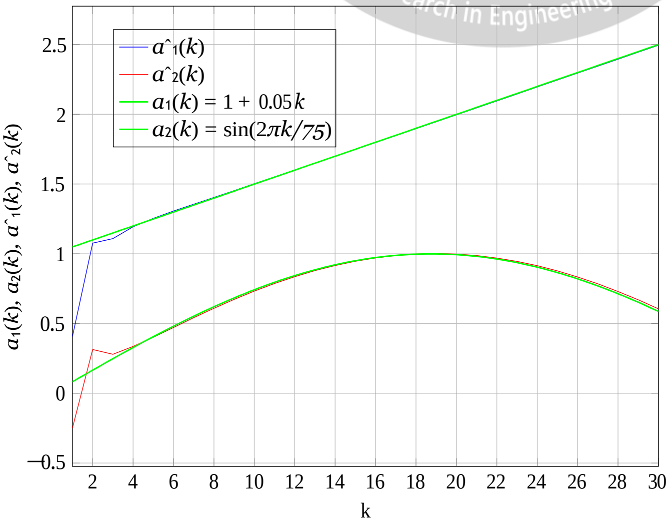


Figure 4: Vector $a(k)$ and its estimate $\hat{a}(k)$

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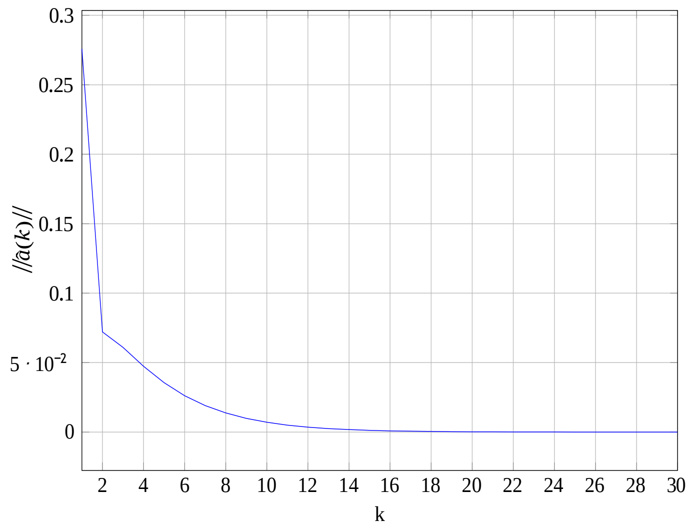


Figure 5 : Norm of the vector $\hat{a}(k)$

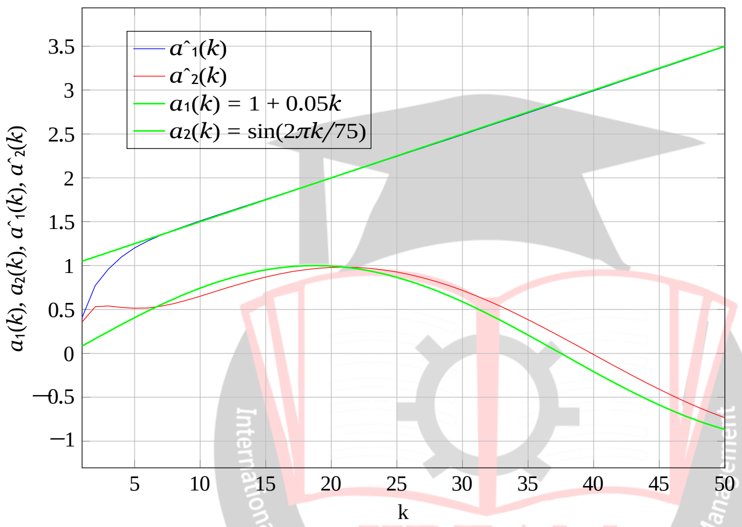


Figure 6 : Vector $a(k)$ and its estimate $\hat{a}(k)$

## III. DEVELOPMENT OF ESTIMATION ALGORITHM

Consider the following difference equation,

$$\hat{\theta}(k+1) = [I - \mu(k)v(t)v^T(k)]\hat{\theta}(k) + \mu(k)v(t)v^T(k)\theta \quad (4)$$

where $\theta$ is an unknown constant vector and $v(k)$ is a nonzero vector that contains known quantities. The two vectors are chosen so that their inner product is the output of a given system at the time instant k+1, i.e., $y(k+1) = v^T(k)\theta$, and $\mu(k)$ is an arbitrary coefficient which can be either a time-varying or a constant. Thus, Eq. (4) can be rewritten as

$$\hat{\theta}(k+1) = [I - \mu(k)v(t)v^T(k)]\hat{\theta}(k) + \mu(k)v(t)y(k+1) \quad (5)$$

Eq. (5) shows that the output $y(k)$ of a system is incorporated in estimating the unknown vector $\theta$. Let $\Gamma(k) = v(k)v^T(k)$, then Eq. (4) becomes

$$\hat{\theta}(k+1) = [I - \mu(k)\Gamma(k)]\hat{\theta}(k) + \mu(k)\Gamma(k)\theta \quad (6)$$

According to Theorem 1, in order for $\hat{\theta} \to \theta$ as $k \to \infty$ in Eq. (4), the time response of the homogeneous

$$\hat{\theta}(k+1) = [I - \mu(k)\Gamma(k)]\hat{\theta}(k)$$

has to approach zero for any $\hat{\theta}_0$ as $k \to \infty$. Since the matrix $\Gamma$ is defined as $\Gamma(k) = v(k)v^T(k)$, the matrix $(I - \mu(k)v(t)v^T(k))$ in Eq. (4) will always have an eigenvalue of value one (Spectral Theorem), which will also make its norm have a value of at least one, therefore it cannot be be made to satisfy Theorem 2. As a result, we will resort to linear algebra techniques that satisfy Condition (1) of

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Theorem 1. This will be the topic of the next section.

**Estimation Algorithm Building Blocks**

**Theorem 3.** *For any k, the matrix* $\Gamma(k) = v(k)v^T(k)$ *is singular with* $n-1$ *zero eigenvalues and one eigenvalue equal to the inner product* $v(k)v^T(k)$. *The vector v(k) is an eigenvector of* $\Gamma(k)$ *for the eigenvector* $v(k)v^T(k)$.

***Proof.*** We omit the index $k$ for notation convenience. Let $\lambda$ and $r$ be and eigenvalue and eigenvector of the matrix $\Gamma = vv^T$ respectively,

$$\Gamma r = \lambda r \text{ or } vv^T r = \lambda r \qquad (7)$$

If $v^T r = 0$, then $\lambda$ must be zero. If on the other hand $v^T r \neq 0$ and $\lambda \neq 0$, then

$$r = \frac{v^T r}{\lambda} v$$

Thus, the vector *r* is a scalar multiple of the vector *v*. Substituting for *r* in Eq. (7) gives

$$\frac{v^T r}{\lambda} vv^T v = \frac{v^T r}{\lambda} \lambda v,$$

which yields

$$(v^T v)v = \lambda v.$$

It maybe concluded that the only nonzero eigenvalue of $\Gamma$ is $\lambda = v^T v$. Since the vector *r* is a scalar multiple of vector *v*, *v* must be the unique eigenvector of the matrix $\Gamma$ for $\lambda$. Let $w_l, l = 1, \dots, n-1$, be a basis of the subspace given by $v^T w_l = 0$; then $w_l$ are the eigenvectors for the eigenvalue zero of multiplicity $n-1$.

The following Theorem is standard [3]. Recall that for an analytic function *f* and a square matrix *A*, *f* (*A*) is defined by substituting *A* into the power series of *f* .

**Theorem 4.** *(Spectral Theorem) For an analytic function f and a square matrix A, the eigenvalues of f (A) are exactly f (λ) for all λ of A.*

**Theorem 5.** *Consider the homogeneous equation*

$$\hat{\theta}(k+1) = [I - \mu(k)v(k)v^T(k)]\hat{\theta}(k) \qquad (8)$$

*associated with Eq. (4) and assume* $\mu(k)v^T(k)v(k) \in (0,2)$, *then*

$$||\hat{\theta}(k+1)||_2 \leq ||\hat{\theta}(k)||_2. \qquad (9)$$

***Proof.*** From Eq. (8)

$$||\hat{\theta}(k+1)||_2 = ||[I - \mu(k)v(k)v^T(k)]\hat{\theta}(k)||_2$$

which implies

$$||\hat{\theta}(k+1)||_2 \leq ||[I - \mu(k)v(k)v^T(k)]||_2 ||\hat{\theta}(k)||_2.$$

But,

$$||[I - \mu(k)v(k)v^T(k)]\hat{\theta}(k)||_2 = \sqrt{\rho([I - \mu(k)v(k)v^T(k)]^T[I - \mu(k)v(k)v^T(k)])}$$

where $\rho$ denotes the spectral radius of a matrix. Let $\alpha(k) = \mu(k)v(k)v^T(k)$, then

$$\rho = \max_{\lambda_i}[I - 2\mu(k)v(k)v^T(k) + \alpha(k)\mu(k)v(k)v^T(k)]$$

$$\rho = \max_{\lambda_i}[I - 2\frac{\alpha(k)}{v^T(k)v(k)}v(k)v^T(k) + \frac{\alpha^2(k)}{v^T(k)v(k)}v(k)v^T(k)]$$

$\rho = \max_{\lambda_i} f(A)$, where $A = \frac{1}{v^T(k)v(k)} v(k)v^T(k)v(k)$ with eigenvalues given by Theorem 3 as $\lambda_1 = 1, \lambda_2 = \lambda_3 = \cdots = \lambda_n = 0$, and $f(\lambda) = 1 - 2\alpha(k)\lambda + \alpha^2(k)\lambda^2(k)$. Theorem 4 gives,

$$\rho = \max_{\lambda_i}(1 - 2\alpha(k)\lambda_i + \alpha^2(k)\lambda_i^2) = \left(1 - \alpha(k)\right)^2.$$

Since $\alpha(k) \epsilon (0,2)$, the proof is concluded. □

The next question to consider is when Eq. (9) has a strict inequality, i.e.,

$$||\hat{\theta}(k+1)||_2 < ||\hat{\theta}(k)||_2.$$

To answer this, the following Theorem is required.

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**Theorem 6.** *Given any two nonzero n-order vectors v and x,*

$$||(I - vv^T)x|| = ||x||$$

*if and only if* $v^Tx = 0$ *or* $v^Tv = 2$.

***Proof.*** For the vectors *v* and *x* there always exists a vector *w* and a nonzero real constant $\xi$ such that

$$x = \zeta v + w, \quad v^Tw = 0.$$

Then

$$||(I - vv^T)x|| = ||x||$$

Is equivalent to

$$||(I - vv^T)(\zeta v + w)|| = ||\zeta v + w||$$

which is equivalent to, since $v^Tw = 0$,

$$||(I - vv^T)\zeta v + w|| = ||\zeta v + w||$$

or

$$||\zeta(1 - v^Tv)v + w|| = ||\zeta v + w||$$

Using $v^Tw = 0$ again gives that the last equation is again equivalent to

$$||\zeta(1 - v^Tv)v||^2 + ||w||^2 = ||\zeta v||^2 + ||w||^2$$

or

$$||\zeta(1 - v^Tv)v||^2 = ||\zeta v||^2$$

or

$$|\zeta(1 - v^Tv)| \, ||v|| = |\zeta| \, ||v||.$$

Because $v \neq 0$, this is equivalent to

$$|\zeta(1 - v^Tv)| = |\zeta|.$$

This is true only if $\xi = 0$ or $|\zeta(1 - v^Tv)| = 1$. But $v^Tv \neq 0$, concluding the proof. □

The next Theorem offers a solution to the problem of making the time response of the homogeneous equation associated with Eq. (4) go to zero.

**Theorem 7.** *Let*

$$\hat{\theta}(k+1) = [I - \mu(k)v(k)v^T(k)]\hat{\theta}(k)$$

*where v(k) are nonzero vectors and* $\mu(k)v(k)v^T(k) = \alpha(k) \epsilon (0,2)$. *If* $\hat{\theta}(k) \neq 0$ *and there are n linearly independent vectors v(i) between the vectors* $v(k), \ldots, v(k+m)$, *then*

$$||\hat{\theta}(k+m)||_2 < ||\hat{\theta}(k)||_2$$

***Proof.*** Assume $\hat{\theta}(k) \neq 0$ and

$$||\hat{\theta}(k+m)||_2 = ||\hat{\theta}(k)||_2$$

Then it follows from Eq. (9) that

$$||\hat{\theta}(k+m)||_2 =, \ldots, ||\hat{\theta}(k+1)||_2 = ||\hat{\theta}(k)||_2$$

Thus

$$||\big(I - \mu(k)v(k)v^T(k)\big)\hat{\theta}(k)||_2 = ||\hat{\theta}(k)||_2$$

Since $\alpha(k) \epsilon (0,2)$, then from Theorem 6 it follows that

$$v^T(k)\hat{\theta}(k) = 0$$

or

$$\hat{\theta}(k+1) = \hat{\theta}(k).$$

Similarly

$$||\hat{\theta}(k+2)||_2 = ||\hat{\theta}(k+1)||_2$$

or

$$||\big(I - \mu(k+1)v(k+1)v^T(k+1)\big)\big(I - \mu(k)v(k)v^T(k)\big)\hat{\theta}(k)||_2 = ||\hat{\theta}(k)||_2$$

or

$$||\big(I - \mu(k+1)v(k+1)v^T(k+1)\big)\hat{\theta}(k)||_2 = ||\hat{\theta}(k)||_2$$


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and again, this means

$v^T(k+1)\hat{\theta}(k) = 0.$

By induction, it follows that

$\hat{\theta}(k+m) = \cdots = \hat{\theta}(k+1) = \hat{\theta}(k)$

and thus

$v^T(i)\hat{\theta}(k) = 0$

for all $i = k, \dots, k+m$. Because the vectors $v(i), i = k, \dots, k+m$, span the space of all n-dimensional vectors, this means that $\hat{\theta}(k) = 0$ , which is a contradiction. □

The above results and Eq. (4) suggest the following: if the vector $v(k)$ varies with time sufficiently enough, then there is some number *m* in which there are an *n* linearly independent vectors $v(k)$ $(k = 1,2, \dots)$. Therefore according to Theorem 7 the norm of the time response of $\hat{\theta}(k)$ of the homogeneous equation associated with Eq. (4) becomes smaller each time *m* occurs. As *k* grows indefinitely, the number *m* repeats a number of times hopefully decreasing the time response of $\hat{\theta}(k)$ to zero, therefore the condition stated in the introduction is satisfied and the proposed algorithm guarantees con- vergence of $\hat{\theta}(k)$. It is also worth noting that if the sequence of vectors $v(k)$ in Eq. (4) are orthogonal and if $\alpha(k) = 1$, then the time response of $\hat{\theta}(k)$ of the homogeneous equation will go to zero in *n* iterations where *n* is the dimension of the vectors $v(k)$.

## IV. FINAL FORM OF THE PROPOSED ALGORITHM

Rearranging Eq. (5) terms yields

$$\hat{\theta}(k+1) = \hat{\theta}(k) + \frac{\alpha(k)v(k)}{v^T(k)v(k)}[y(k+1) - v^T(k)\hat{\theta}(k) \quad (10)$$

Defining $K(k+1) = \frac{\alpha(k)v(k)}{v^T(k)v(k)}$, we have

$$\hat{\theta}(k+1) = \hat{\theta}(k) + K(k+1)[y(k+1) - v^T(k)\hat{\theta}(k) \quad (11)$$

Eq. (11) has the same general form as those of Kalman filtering [4], Recursive Least-Squares [5], and Stochastic approximation [6]. The only difference is the gain vector *K*(*k*). Table 1 shows that they are all linear in form. Depending on the formulation of the problem, all algorithms can be used to either identify system parameters or observe system states, see [7,8,9].

| Algorithm | $K(k+1)$ | $P(k+1)$ |
|---|---|---|
| Kalman | $\frac{P(k)v^T(k)}{v(k)P(k)v^T(k)+\hat{\eta}}$ | $[I - K(k+1)v^T(k)]P(k)$ |
| Recursive exponentially weighted least-squares | $\frac{P(k)v^T(k)}{v(k)P(k)v^T(k)+\gamma}$ | $[I - K(k+1)v^T(k)]P(k)/\gamma$ |
| Stochastic approximation | $\frac{1}{1+k}\frac{v(k)}{\|\|v(k)\|\|_2^2}$ | -------------- |
| Proposed algorithm | $\frac{1}{1+k}\frac{v(k)}{\|\|v(k)\|\|_2^2}$ | -------------- |

Table 1: Comparison of Estimation Algorithms.

**Error of The Proposed Algorithm**

Let $e(k+1) = \hat{\theta}(k+1) - \theta$ be the error at time $k+1$, then $\alpha(k) = 1$ gives the minimum least-squares error. This can be seen as follows: from Eq. (4),

$$e(k+1) = \left[I - \frac{\alpha(k)v(k)}{v^T(k)v(k)}v^T(k)\right]e(k)$$

where $e(k) = \hat{\theta}(k) - \theta$. The least-squares error is given by


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$$e^T(k+1)e(k+1) = \left[e(k) - \frac{\alpha(k)v(k)}{v^T(k)v(k)}v^T(k)e(k)\right]^T \left[e(k) - \frac{\alpha(k)v(k)}{v^T(k)v(k)}v^T(k)e(k)\right]$$

$$= e^T(k)e(k) - 2\frac{\alpha(k)}{v^T(k)v(k)}e^T(k)v(k)v^T(k)e(k) + \frac{\alpha^2(k)}{v^T(k)v(k)}e^T(k)v(k)v^T(k)e(k).$$

If $v(k)e(k) \neq 0$, then

$$0 = \frac{\partial e^T(k+1)e(k+1)}{\partial \alpha(k)} = \frac{(\alpha(k)-1)}{v^T(k)v(k)}e^T(k)v(k)v^T(k)e(k).$$

**Stability of The Proposed Algorithm**

The system of Eq. (4) is BIBO (bounded-input bounded-output) for a bounded vector $\theta(k)$. Also, according to reference [2] the homogeneous part of Eq. (4) is stable and if $\hat{\theta}(k) \to 0$ as $k \to \infty$, then it is *asymptotically stable.*

## V. CONCLUSION

A proposed new estimation algorithm was developed in which state space and linear algebra were utilized in its derivation. It was shown that the sequence of vectors $v(k)$ are function of the system input signal $u(k)$, and in order for them to have enough linearly independent vectors, $u(k)$ must be sufficiently rich. Sufficiently rich means that $u(k)$ must vary enough so that the sequence of vectors $v(k)$ cause the homogeneous response of Eq. (4) go to zero.

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