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The Double Fermat Number in Maximum and Minimum Matrices and its Divisibility

Dr. R. Muthamizh Selvi,

Assistant Professor, Department of Mathematics, Saradha Gangadharan Arts & Science College, Puducherry, India. muthamizhmaths@gmail.com.

Abstract - We define the Double Fermat minimum matrices and Double Fermat Maximum matrices separately. Also, we calculate the determinant value and its inverse. Double Fermat numbers are a subset of Fermat numbers, so Double Fermat primes are a subset of Fermat primes. Also, we discuss the divisibility of the Double Fermat Maximum matrices by Double Fermat Minimum matrices.

Keywords - Maximum matrix, Minimum matrix, Double Fermat Maximum matrix and Double Fermat Minimum matrix.

I. INTRODUCTION

Let $(P, \preccurlyeq) = (P, \wedge, \vee)$ be a lattice, let $S = \{x_1, x_2, x_3, \dots \dots, x_n\}$ be a meet-closed subset of P and let $f: P \rightarrow Z^+$ be a function. The minimum matrix $(S)_f$ and the maximum matrix $[S]_f$ on S with respect to f are **[1]** defined by $(S)_f = min\,(x_i, x_j)$ and $[S]_f = max\,[x_i, x_j]$.

It is well known that $(Z_+, |) = (Z_+, min, max)$ is a lattice, where **|** is the usual divisibility relation for the minimum number and the maximum number of integers. Thus minimum and maximum matrices **[2]** are generalizations of Minimum matrices $\left((S)_f\right)_{ij} = min\,(x_i, x_j)$ and Maximum matrices $\left([S]_f\right)_{ij} = max\,[x_i, x_j]$.

We begin by presenting the definition of Minimum matrix. Let $T = \{z_1, z_2, z_3, \dots \dots \dots, z_n\}$ be a finite multiset of real numbers, where $z_1 \le z_2 \le \cdots \dots \dots \le z_n$ (in some cases, however, we need to assume that $z_1 < z_2 < \cdots \dots \dots < z_n$). The Minimum matrix T_{min} of the set T has $min(z_i, z_j)$ as its ij entry. Whereas the Maximum matrix T_{max} of the set T has $max(z_i, z_j)$ as its ij entry. Both the matrices **[4]** are clearly square and symmetric and they may be written explicitly as,

$$T_{min} = \begin{bmatrix} z_1 & z_1 & z_1 & \cdots & z_1 \\ z_1 & z_2 & z_2 & \cdots & z_2 \\ z_1 & z_2 & z_3 & \cdots & z_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_1 & z_2 & z_3 & \cdots & z_n \end{bmatrix}$$

$$T_{max} = \begin{bmatrix} z_1 & z_2 & z_3 & \cdots & z_n \\ z_2 & z_2 & z_3 & \cdots & z_n \\ z_3 & z_3 & z_3 & \cdots & z_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_n & z_n & z_n & \cdots & z_n \end{bmatrix}.$$

II. SOME BASIC DEFINITIONS

In this section we discuss some basic definitions of maximum and minimum matrices.

2.1 Definition

Let $S = \{x_1, x_2, \dots \dots \dots x_n\}$ be a set of distinct positive integers and S be a set of a lattice $(P, \preccurlyeq)$. The Fermat Minimum matrix **[3]** with respect to a complex valued function f is defined by $(S)_f = (f_{ij}),\ where\ f_{ij} = 2^{2^{min\,(x_i, x_j)}} + 1.$

2.2 Definition

Let $S = \{x_1, x_2, \dots \dots \dots x_n\}$ be a set of distinct positive integers and S be a set of a lattice $(P, \preccurlyeq)$. The Fermat Maximum matrix **[3]** with respect to a complex valued function f is defined by $[S]_f = (f_{ij}),\ where\ f_{ij} = 2^{2^{max\,(x_i, x_j)}} + 1.$

2.3 Definition

If F_n is a Fermat number then $FF_n = 2^{F_n - 1} + 1$, is called a Double Fermat number, where n is a natural number.

2.4 Definition

Let $S = \{x_1, x_2, \dots \dots \dots x_n\}$ be a set of distinct positive integers and S be a set of a lattice $(P, \preccurlyeq)$ and the $n \times n$ matrix $(FF_n)_f = (f_{ij})\ where\ f_{ij} = 2^{F_n - 1} + 1$ is called the Double Fermat Minimum Matrix with respect to a complex valued function f on S, and F_n is called Fermat minimum matrix, where, $F_n = 2^{2^{min\,(x_i, x_j)}} + 1.$

2.5 Definition

Let $S = \{x_1, x_2, \dots \dots \dots x_n\}$ be a set of distinct positive integers and S be a set of a lattice $(P, \preccurlyeq)$ and the nxn matrix $[FF_n]_f = (f_{ij})\ where\ f_{ij} = 2^{F_n - 1} + 1$ is called the Double Fermat Maximum Matrix with respect to a complex

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valued function f on S, and F_n is called Fermat maximum matrix, where, $F_n = 2^{2^{max(x_i,x_j)}} + 1$.

III. DETERMINANT VALUE OF DOUBLE FERMAT MINIMUM AND MAXIMUM MATRICES

Let $S = \{x_1, x_2, \dots \dots \dots x_n\}$ be a Minimum closed subset of P. Then the determinant of the Double Fermat Minimum matrix is denoted by $(FF_n)_f$ and defined as,

$$\det (FF_n)_f = f(x_1)\{[f(x_2) - f(x_1)][f(x_3) - f(x_2)] \dots \dots \dots [f(x_n) - f(x_{n-1})]\}.$$

$$\det[FF_n]_f = [f(x_1) - f(x_2)][f(x_2) - f(x_3)] \dots \dots \dots [f(x_{n-1}) - f(x_n)]f(x_n)$$

. where $f(x_i) = 2^{2^{x_i}} + 1 \quad j = 1,2, \dots \dots n.$

IV. INVERSE OF DOUBLE FERMAT MINIMUM AND DOUBLE FERMAT MAXIMUM MATRICES

Suppose that the elements of the set $(S)_f$ are distinct. If $x_1 \neq 0$, then the Double Fermat Minimum matrix is invertible and the inverse matrix is the $n \times n$ tridiagonal matrix $(S)_f^{-1} = R = (r_{ij})$,

Where $(S)_f^{-1} =$

$$\begin{cases} 0 & if\ |i-j| > 1 \\ \frac{f(x_2)}{f(x_1)[f(x_2)-f(x_1)]} & if\ i = j = 1 \\ \frac{1}{f(x_i)-f(x_{i-1})} + \frac{1}{f(x_{i+1})-f(x_i)} & if\ 1 < i = j < n \\ \frac{1}{f(x_n)-f(x_{n-1})} & if\ i = j = n \\ \frac{-1}{|f(x_i)-f(x_j)|} & if\ |i-j| = 1 \end{cases}$$

$f(x_i) = 2^{2^{x_i}} + 1, \quad i = 1,2, \dots \dots n.$

Suppose that the elements of the set $[S]_f$ are distinct. If $x_n \neq 0$, then the S-prime Maximum matrix is invertible and the inverse matrix is the $n \times n$ tridiagonal matrix $[S]_f^{-1} = T = (t_{ij})$,

where, $[S]_f^{-1} =$

$$\begin{cases} 0 & if\ |i-j| > 1 \\ \frac{1}{[f(x_1)-f(x_2)]} & if\ i = j = 1 \\ \frac{1}{f(x_{i-1})-f(x_i)} + \frac{1}{f(x_i)-f(x_{i+1})} & if\ 1 < i = j < n \\ \frac{1}{f(x_{n-1})-f(x_n)} + \frac{1}{f(x_n)} & if\ i = j = n \\ \frac{1}{|f(x_i)-f(x_j)|} & if\ |i-j| = 1. \end{cases}$$

$f(x_i) = 2^{2^{x_i}} + 1, \ i = 1,2, \dots \dots n.$

4.1 Example

If $S = \{1,2,3\}$ is a lower closed set. Construct Double Fermat Minimum and Double Fermat Maximum matrix also find its determinant value.

Solution

By definition of Double Fermat Minimum matrix is,

$$(FF_n)_f = (f_{ij}) \ where\ f_{ij} = 2^{F_n-1} + 1$$

$$(FF_n)_f = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} 5 & 5 & 5 \\ 5 & 17 & 17 \\ 5 & 17 & 257 \end{bmatrix}.$$

The determinant value of the Double Fermat Minimum Matrix is,

$$\det (FF_n)_f = f(x_1)\{[f(x_2) - f(x_1)][f(x_3) - f(x_2)]\}.$$
$$= 14400.$$

The Inverse of the Double Fermat Minimum Matrix is,

$$(S)_f^{-1} = R = (r_{ij}) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$r_{11} = \frac{17}{60}, r_{12} = \frac{1}{12}, r_{13} = 0$

$r_{21} = \frac{1}{12}, r_{22} = \frac{21}{240}, r_{23} = \frac{1}{240}.$

$r_{31} = 0, r_{32} = \frac{1}{240}, r_{33} = \frac{1}{240}.$

$$\therefore \ (FF_n)_f^{-1} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} \frac{17}{60} & \frac{-1}{12} & 0 \\ \frac{-1}{12} & \frac{7}{80} & \frac{-1}{240} \\ 0 & \frac{-1}{240} & \frac{1}{240} \end{bmatrix}.$$

By definition of Double Fermat Maximum Matrix is,

$$[FF_n]_f = (f_{ij}) \ where\ f_{ij} = 2^{F_n-1} + 1$$

$$[FF_n]_f = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} 5 & 17 & 257 \\ 17 & 17 & 257 \\ 257 & 257 & 257 \end{bmatrix}.$$

The determinant value of the Double Fermat Maximum Matrix is,

$$\det [FF_n]_f = \{[f(x_1) - f(x_2)][f(x_2) - f(x_3)]\ f(x_3)\}.$$
$$= 740160.$$

The Inverse of the Double Fermat Maximum Matrix is,

$$[S]_f^{-1} = T = (t_{ij}) = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix}$$

$t_{11} = \frac{-1}{12}, t_{12} = \frac{1}{12}, t_{13} = 0$

$t_{21} = \frac{1}{12}, t_{22} = \frac{-21}{240}, t_{23} = \frac{1}{240}.$

$t_{31} = 0, t_{32} = \frac{1}{240}, t_{33} = \frac{-17}{61680}.$

$$\therefore \ [FF_n]_f^{-1} = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix} = \begin{bmatrix} \frac{-1}{12} & \frac{1}{12} & 0 \\ \frac{1}{12} & \frac{-7}{80} & \frac{1}{240} \\ 0 & \frac{1}{240} & \frac{-17}{61680} \end{bmatrix}.$$

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V. Divisibility of Double Fermat Maximum Matrices by Double Fermat Minimum Matrices

Let S be a minimum-closed or maximum-closed set with n elements, where $n \leq 3$. Let f be a semi multiplicative function satisfying $f(x_i) \neq 0$ for all $x_i, x_j \in S$. Then $[S]_f | (S)_f$.

Proof:

Suppose first that S is a gcd-closed set with n elements. If $n = 1$, then $(S)_f = [S]_f$. Let $n = 2$. Then $x_1 | x_2$ and thus according to (2.2) we have $f(x_1) | f(x_2)$ and further

$$[S]_f(S)_f^{-1} = \begin{bmatrix} f(x_1) & f(x_2) \\ f(x_2) & f(x_2) \end{bmatrix} \begin{bmatrix} f(x_1) & f(x_1) \\ f(x_1) & f(x_2) \end{bmatrix}^{-1}$$
$$= \begin{bmatrix} 0 & 1 \\ \frac{f(x_2)}{f(x_1)} & 0 \end{bmatrix} \in \mathcal{M}_3(\mathbb{Z})$$

Let n = 3. Then either $x_1 | x_2 | x_3 \ or (x_2, x_3) = x_2.$ Let $x_1 | x_2 | x_3$. Then according to (2.7) we have $f(x_1) | f(x_2) | f(x_3)$ and further

$$[S]_f(S)_f^{-1} =$$
$$\begin{bmatrix} f(x_1) & f(x_2) & f(x_3) \\ f(x_2) & f(x_2) & f(x_3) \\ f(x_3) & f(x_3) & f(x_3) \end{bmatrix} \begin{bmatrix} f(x_1) & f(x_1) & f(x_1) \\ f(x_1) & f(x_2) & f(x_2) \\ f(x_1) & f(x_2) & f(x_3) \end{bmatrix}^{-1}$$
$$= \begin{bmatrix} 0 & 0 & 1 \\ \frac{f(x_2)}{f(x_1)} & -1 & 1 \\ \frac{f(x_3)}{f(x_1)} & 0 & 0 \end{bmatrix} \in \mathcal{M}_3(\mathbb{Z})$$

The same result is also true for Double Fermat Minimum Matrices and Double Fermat Maximum Matrices.

5.1 Example

Let $S = \{1,2,3\}$ is a lower closed set. By definition of Double Fermat Minimum matrix is,

$$(FF_n)_f = \begin{bmatrix} 5 & 5 & 5 \\ 5 & 17 & 17 \\ 5 & 17 & 257 \end{bmatrix},$$
$$[FF_n]_f = \begin{bmatrix} 5 & 17 & 257 \\ 17 & 17 & 257 \\ 257 & 257 & 257 \end{bmatrix} \ and$$
$$(FF_n)_f^{-1} = \begin{bmatrix} \frac{17}{60} & \frac{-1}{12} & 0 \\ \frac{-1}{12} & \frac{7}{80} & \frac{-1}{240} \\ 0 & \frac{-1}{240} & \frac{1}{240} \end{bmatrix}$$

Then the Divisibility of Double Fermat Maximum Matrices by Double Fermat Minimum Matrices is,

$$[FF_n]_f | (FF_n)_f = [FF_n]_f (FF_n)_f^{-1}$$
$$= \begin{bmatrix} 0 & 0 & 1 \\ \frac{17}{5} & -1 & 1 \\ \frac{257}{5} & 0 & 0 \end{bmatrix}.$$

VI. Characteristics of Double Fermat Matrices

1. If double Fermat matrices is prime then the double Fermat matrices is called the double Fermat prime matrices.
2. All the natural numbers are called the basic sequence of number of double Fermat prime numbers.
3. In the first few continuous natural numbers n make $2^{F_n - 1} + 1$ become double Fermat prime matrices in basic sequence of number of double Fermat prime matrices then these natural numbers are called original continuous natural number sequence of double Fermat prime matrices.
4. The original continuous natural number sequence of double Fermat primes is $n = 0,1,2.$
5. Double Fermat prime matrices are strongly finite if the first few continuous terms generated from the original continuous natural number sequence are prime but all larger terms are composite number.
6. Double Fermat primes are infinite if both the sum of corresponding original continuous natural number sequence and the first such prime are Fermat prime matrices, but such prime matrices are strongly finite if one of them is not Fermat prime matrices, then double Fermat prime matrices are strongly finite.

VII. Conclusion

In this paper, the different properties of Double Fermat Minimum and Double Fermat Maximum matrices of the set S with $min\,(x_i, x_j)$ and $max\,(x_i, x_j)$ as their (i, j) entries like determinant value and inverse of Double Fermat Minimum and Double Fermat Maximum matrices have been studied. The study is carried out by applying known results of meet and joins matrices to Fermat minimum and Fermat maximum matrices.

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