International Journal for Research in Engineering Application & Management (IJREAM)
ISSN : 2454-9150 Vol-12, Issue-04, July 2026

First-Order Normalization in the Generalized Photogravitational Restricted Three-Body Problem

Mahesh Kumar[1], Avdhesh Kumar[2]

[1]Research Scholar, Department of Mathematics, J.P. University, Chapra, Bihar, India.

[2]Assistant Professor, Department of Mathematics, Jaglal Chaudhary College, Chapra (A Constituent unit of J.P. University, Chapra), Bihar, India.

ABSTRACT - The restricted three-body problem is a fundamental model in celestial mechanics. In this paper study, we extend it to the generalized photogravitational restricted three-body problem, including radiation pressure and oblateness of the primaries. Using a rotating coordinate system, the Lagrangian is expanded around the triangular equilibrium point L_4. The Hamiltonian is then simplified into normal form through canonical transformations. Results show that radiation pressure and oblateness perturb the basic frequencies, influencing the stability of triangular points. This normalization provides a clearer Hamiltonian structure, useful for analyzing periodic solutions and spacecraft trajectories near Lagrangian points.

Keywords- RTBP, Photogravitational Effects, Hamiltonian Normalization, Lagrangian Equilibrium Points, Space Dynamics.

I. INTRODUCTION

The study of planetary motion has fascinated scientists for centuries, beginning with Newton's laws and later expanded by Euler, Lagrange, Laplace, and Poincaré. While the two-body problem has exact solutions, the three-body problem remains complex and unsolved in closed form. To make progress, researchers often simplify it into the restricted three-body problem (RTBP), where a small body of negligible mass moves under the gravitational influence of two larger primaries without affecting their motion.

Within this framework, Lagrange identified five equilibrium points, now known as Lagrangian points (L1–L5). These points are critical in celestial mechanics and space exploration, as they represent positions where gravitational and centrifugal forces balance. The triangular points (L4 and L5) can be stable under certain mass ratios, and they host natural objects such as Trojan asteroids, as well as artificial satellites like SOHO and JWST.

Modern extensions of the RTBP include the photogravitational restricted three-body problem, which accounts for radiation pressure from luminous primaries, and the oblateness effect, which considers the flattening of massive bodies. These refinements make the model more realistic for astrophysical and space mission applications.

The aim of this work is to perform a first-order normalization of the Hamiltonian in the generalized photogravitational RTBP. By expanding the Lagrangian around the triangular point L_4 and applying canonical transformations, we analyze how radiation pressure and oblateness perturb the basic frequencies of motion. This approach provides deeper insight into the stability of equilibrium points and offers practical applications in space dynamics, particularly for spacecraft trajectory design near Lagrangian regions.

Lagrangian Equilibrium Points

In the framework of a rotating coordinate system, Joseph-Louis Lagrange identified five special positions where a small body of negligible mass can remain in balance under the combined gravitational pull of two larger orbiting bodies. These positions are known as Lagrangian points (L1–L5). At these points, the gravitational attraction of the two primaries is exactly balanced by the centripetal force required for the small body to revolve with them, making them equilibrium points.

Three of these points lie along the line joining the two massive bodies and are called collinear points (L1, L2, L3).

L1 lies between the two bodies and provides uninterrupted views of the Sun, making it the location of the SOHO satellite. L2 lies just beyond the smaller body and hosts missions such as WMAP and the James Webb Space Telescope (JWST). L3 lies on the far side of the larger body, hidden behind the Sun, and is considered impractical for space missions.

The other two points, L4 and L5, form equilateral triangles with the two primaries and are called triangular points. These points can be stable if the mass ratio of the primaries is below a certain threshold. In the Earth–Sun and Earth–Moon systems, as well as in many other celestial pairs, L4 and L5 host stable orbits. Objects located here are known as Trojans. Numerous Trojan asteroids orbit with Jupiter and Mars, and even Saturn's moons have Trojan companions. In 1956, Polish astronomer Kordylewski reported concentrations of dust near the Earth–Moon Trojan points.

DOI : 10.35291/2454-9150.2026.0037

ISSN : 2454-9150 Vol-12, Issue-04, July 2026

The stability and existence of these points have been studied extensively by researchers such as Markeev and Sokolski (1977), Singh et al. (2005), Singh and Omale (2019), and others. In more advanced models, the photogravitational restricted three-body problem extends the classical case by including radiation pressure from luminous primaries, while oblateness accounts for the flattening of massive bodies. These refinements alter the stability and exact positions of the Lagrangian points, making the model more realistic for astrophysical applications and space mission planning.

II. EXPANSION OF LAGRANGIAN

Using dimensionless variables and a synodic coordinate system (x,y) as Szebehely (1967) and Singh and Ishwar (1999) the equations of motion can be written as

$$\ddot{x} - 2n\dot{y} = U_x$$

$$\ddot{y} + 2n\dot{x} = U_y \qquad (4.2.1)$$

where

$$U = \frac{1}{2}n^2\left(x^2 + y^2\right) + \frac{(1-\mu)q_1}{r_1} + \frac{\mu q_2}{r_2} + \frac{A_1 q_1 (1-\mu)}{2r_1^3} + \frac{A_2 q_2 \mu}{2r_2^3} \qquad (4.2.2)$$

$$r_1^2 = (x-\mu)^2 + y^2$$

$$r_2^2 = (x+1-\mu)^2 + y^2$$

$$\mu = \frac{m_2}{(m_1 + m_2)} \leq \frac{1}{2}$$

$m_1, m_2\ (m_1 \geq m_2)$ are masses of the primaries. The perturbed mean motion of the primaries n is given by

$$n^2 = 1 + \frac{3}{2}(A_1 + A_2)$$

The Lagrangian function of the problem can be written as

$$L = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) + n(x\dot{y} - \dot{x}y) + \frac{n^2}{2}\left(x^2 + y^2\right)$$
$$+ \frac{(1-\mu)q_1}{r_1} + \frac{\mu q_2}{r_2} + \frac{A_1 q_1 (1-\mu)}{2r_1^3} + \frac{A_2 q_2 \mu}{2r_2^3} \qquad (4.2.3)$$

The Hamiltonian H of the problem may be written as

$$H = -L + P_x\dot{x} + P_y\dot{y}$$

$$= \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) - \frac{n^2}{2}\left(x^2 + y^2\right) - \frac{(1-\mu)q_1}{r_1} - \frac{\mu q_2}{r_2} - \frac{A_1 q_1 (1-\mu)}{2r_1^3} - \frac{A_2 q_2 \mu}{2r_2^3}$$

$$= \frac{1}{2}\left(P_x^2 + P_y^2\right) + n\left(yP_x - xP_y\right) - \frac{(1-\mu)q_1}{r_1} - \frac{\mu q_2}{r_2} - \frac{A_1 q_1 (1-\mu)}{2r_1^3} - \frac{A_2 q_2 \mu}{2r_2^3} \qquad (4.2.4)$$

where P_x, P_y are the momenta coordinates given by

$$P_x = \frac{\partial L}{\partial \dot{x}} = \dot{x} - ny$$

DOI : 10.35291/2454-9150.2026.0037

$$P_y = \frac{\partial L}{\partial \dot{y}} = \dot{y} + nx \tag{4.2.5}$$

The coordinates of the triangular points L_4 are

$$x_0 = -\frac{1}{2}\left[\gamma + \frac{2}{3}(q_1 - q_2) + A_1 q_1 - A_2 q_2\right] \quad \text{with} \qquad \gamma = 1 - 2\mu$$

$$y_0 = \frac{\sqrt{3}}{2}\left[1 - \frac{2}{3}(A_1 + A_2) + \frac{1}{3}(A_1 q_1 + A_2 q_2) - \frac{2}{9}(1 - q_1) - \frac{2}{9}(1 - q_2)\right] \tag{4.2.6}$$

We shift the origin to L_4 For what, we change

$$x \to x_0 + x$$

$$y \to y_0 + y$$

The Lagrangian function L becomes

$$L = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) + n\left\{(x_0 + x)\dot{y} - (y_0 + y)\dot{x}\right\} + \frac{n^2}{2}\left\{(x_0 + x)^2 + (y_0 + y)^2\right\}$$

$$+ \frac{(1-\mu)q_1}{r_1} + \frac{\mu q_2}{r_2} + \frac{A_1 q_1 (1-\mu)}{2r_1^3} + \frac{A_2 q_2 \mu}{2r_2^3} \tag{4.2.7}$$

where

$$r_1^{-1} = [(x+a)^2 + (y+b)^2]^{-\frac{1}{2}} = f_1(x,y)$$

$$r_2^{-1} = [(x+a+1)^2 + (y+b)^2]^{-\frac{1}{2}} = f_2(x,y)$$

$$r_1^{-3} = [(x+a)^2 + (y+b)^2]^{-\frac{3}{2}} = f_3(x,y)$$

$$r_2^{-3} = [(x+a+1)^2 + (y+b)^2]^{-\frac{3}{2}} = f_4(x,y)$$

$$a = x_0 + \frac{\gamma - 1}{2}$$

$$b = y_0$$

Now we expand L in power series of x and y.

First of all

We expand $f_i(x,y)$ $i = 1,2,3$

By Taylor's theorem,

$$f_i(x,y) = f_i(0,0) + [x f_{ix}(0,0) + y f_{iy}(0,0)]$$

$$+ \frac{1}{2i}[x^2 + f_{ixx}(0,0) + 2xy\, f_{ixy}(0,0) + y^2 f_{iyy}(0,0)]$$

 DOI : 10.35291/2454-9150.2026.0037

$$+\frac{1}{3i}[x^3+f_{ixxx}(0,0)+3x^2y\,f_{ixxy}(0,0)+3xy^2f_{ixyy}(0,0)+y^3f_{iyyy}(0,0)]$$

$$+\frac{1}{4i}\left[x^4+f_{ixxxx}(0,0)+4x^3y\,f_{ixxxy}(0,0)+6x^2y^2f_{ixxyy}(0,0)\right.$$

$$\left.+4xy^3f_{ixyyy}(0,0)+y^4f_{iyyyy}(0,0)\right]+\ldots\ldots\ldots\ldots,\ \text{for}\ i=1,2,3$$

We have

$$f_{1x}=-(x+a)f_1^3$$

$$f_{1xx}=-f_1^3+3f_1^5(x+a)^2$$

$$f_{1xxx}=9(x+a)f_1^5-15(x+a)^2f_1^7$$

$$f_{1xxxx}=9f_1^5-90f_1^7(x+a)^2+105(x+a)f_1^9$$

$$f_{1y}=-(y+a)f_1^3$$

$$f_{1yy}=(y+b)^2f_1^5-f_1^3$$

$$f_{1yyy}=9(y+b)f_1^5-15f_1^7(y+b)^3$$

$$f_{1yyyy}=9f_1^5-90f_1^7(y+b)^2+105(y+b)^4f_1^9$$

$$f_{1xy}=3(x+a)(y+b)f_1^5$$

$$f_{1xxy}=-15(x+a)^2(y+b)f_1^7+3(y+b)f_1^5$$

$$f_{1xyy}=3(x+a)f_1^5-15(x+a)(y+b)^2f_1^7$$

$$f_{1xxxy}=-45(x+a)(y+b)f_1^7+105(x+a)^3(y+b)f_1^9$$

$$f_{1xxyy}=-15(x+a)^2f_1^7+105(x+a)^2(y+b)^2f_1^9+3f_1^5-15(y+b)^2f_1^7$$

$$f_{1xyyy}=-15(x+a)(y+b)f_1^7-30(x+a)(y+b)f_1^7+105(x+a)(y+b)^3f_1^9$$

$$=45(x+a)(y+b)f_1^7+105(x+a)(y+b)^3f_1^9$$

$$f_{3y}=3f_1^2f_{1y}$$

$$f_{3yy}=6f_1f_{1y}^2+3f_1^2f_{1yy}$$

$$f_{3yyy}=6f_{1y}^3+12f_1f_{1yy}+6f_1f_{1y}f_{1yy}+3f_1^2f_{1yyy}$$

$$f_{3yyyy}=24f_{1y}^2f_{1yy}+12f_{1y}f_{1yy}+12f_1f_{1yyy}+6f_1f_{1yy}^2+6f_1f_{1y}f_{1yyy}$$

$$f_{3xy}=6f_1f_{1y}f_{1x}+3f_1^2f_{1xy}$$

 DOI : 10.35291/2454-9150.2026.0037

$$f_{3xxy} = 3f_{1y}f_{1x}^2 + 12f_1f_{1x}f_{1xy} + 6f_1f_{1y}f_{1xx} + 3f_1^2f_{1xxy}$$

$$f_{3xyy} = 6f_{1y}^2f_{1x} + 6f_1f_{1yy}f_{1x} + 6f_1f_{1y}f_{1xy} + 6f_1f_{1y}f_{1xy} + 3f_{1xyy}^2$$

$$f_{3xxxy} = 18f_{1x}^2f_{1xy} + 12f_{1y}f_{1xx} + 12f_1f_{1xxy} + 6f_{1y}f_{1x}f_{1xx} + 6f_1f_{1xy}f_{1xx}$$
$$+ 6f_1f_{1x}f_{1xxy} + 6f_1f_{1y}f_{1xxx} + 6f_1^2f_{1xxxy}$$

$$f_1(0,0) = (a^2+b^2)^{-\frac{1}{2}}$$

$$f_{1x}(0,0) = -a(a^2+b^2)^{-\frac{3}{2}}$$

$$f_{1xx}(0,0) = -(a^2+b^2)^{-\frac{3}{2}} + 3(a^2+b^2)^{-\frac{5}{2}}a^2$$

$$f_{1xxx}(0,0) = 9a(a^2+b^2)^{-\frac{5}{2}} - 15a^3(a^2+b^2)^{-\frac{7}{2}}$$

$$f_{1xxxx}(0,0) = 9(a^2+b^2)^{-\frac{5}{2}} - 90a^2(a^2+b^2)^{-\frac{7}{2}} + 105a^4(a^2+b^2)^{-\frac{9}{2}}$$

$$f_{1y}(0,0) = -b(a^2+b^2)^{-\frac{3}{2}}$$

$$f_{1yy}(0,0) = (a^2+b^2)^{-\frac{3}{2}} + \left[3b^2\left(a^2+b^2\right)^{-1} - 1\right]$$

$$f_{2yy}(0,0) = \left\{(a+1)^2+b^2\right\}^{-\frac{3}{2}}\left[3b^2\left\{(a+1)^2+b^2\right\}^{-1} - 1\right]$$

$$f_{3yy}(0,0) = 3(a^2+b^2)^{-\frac{5}{2}}\left[5b^2\left(a^2+b^2\right)^{-1} - 1\right] \qquad (4.2.8)$$

$$f_{1yyy}(0,0) = 9b\left(a^2+b^2\right)^{-\frac{5}{2}} - 15\left(a^2+b^2\right)^{-\frac{7}{2}}b^3$$

$$f_{1yyyy}(0,0) = 9\left(a^2+b^2\right)^{-\frac{5}{2}} - 90b^2\left(a^2+b^2\right)^{-\frac{7}{2}} + 105b^4\left(a^2+b^2\right)^{-\frac{9}{2}}$$

$$f_{1xxy}(0,0) = -15a^2b\left(a^2+b^2\right)^{-\frac{7}{2}} + 3b\left(a^2+b^2\right)^{-\frac{5}{2}}$$

$$f_{1xyy}(0,0) = 3a\left(a^2+b^2\right)^{-\frac{5}{2}} - 15ab^2\left(a^2+b^2\right)^{-\frac{7}{2}}$$

$$f_{1xxxy}(0,0) = -45ab\left(a^2+b^2\right)^{-\frac{7}{2}} + 105a^3b\left(a^2+b^2\right)^{-\frac{9}{2}}$$

DOI : 10.35291/2454-9150.2026.0037

$$f_{1xxyy}(0,0)=-15a^2\left(a^2+b^2\right)^{-\frac{7}{2}}+105a^2b^2\left(a^2+b^2\right)^{-\frac{9}{2}}$$

$$+3\left(a^2+b^2\right)^{-\frac{5}{2}}-15b^2\left(a^2+b^2\right)^{-\frac{11}{2}}$$

$$f_{1xyyy}(0,0)=-45ab\left(a^2+b^2\right)^{-\frac{7}{2}}+105ab^3\left(a^2+b^2\right)^{-\frac{9}{2}}$$

$$f_{3xxxx}(0,0)=45\left(a^2+b^2\right)^{-\frac{7}{2}}-630a^2\left(a^2+b^2\right)^{-\frac{9}{2}}+945a^4\left(a^2+b^2\right)^{-\frac{11}{2}}$$

$$f_{3xxxy}(0,0)=-315ab\left(a^2+b^2\right)^{-\frac{9}{2}}+945a^3b\left(a^2+b^2\right)^{-\frac{11}{2}}$$

$$f_{3xyyy}(0,0)=-315ab\left(a^2+b^2\right)^{-\frac{9}{2}}+945ab^3\left(a^2+b^2\right)^{-\frac{11}{2}}$$

$$f_{3xxyy}(0,0)=-90\left(a^2+b^2\right)^{-\frac{7}{2}}+945a^2b^2\left(a^2+b^2\right)^{-\frac{11}{2}}$$

$$f_{3yyyy}(0,0)=45\left(a^2+b^2\right)^{-\frac{7}{2}}-630b^2\left(a^2+b^2\right)^{-\frac{9}{2}}+945b^4\left(a^2+b^2\right)^{-\frac{11}{2}}$$

In order to find the derivatives of f_2 and f_4, we only change a to $a+1$ in similar types of the derivatives of f_1 and f_3 respectively. Substituting the values of r_1^{-1}, r_2^{-1}, r_1^{-3} and r_2^{-3} in (4.2.7) and arranging the terms in ascending powers of x, y we have

$$L_2=\frac{1}{2}\left(\dot{x}^2+\dot{y}^2\right)+n\left(x\dot{y}-\dot{x}y\right)+\frac{n^2}{2}\left(x\ \ +y^2\right)-Ex^2-Gxy-Fy^2$$

$$L_3=\frac{1}{3}T_1x^3+\frac{1}{2}T_2x^2y+T_3xy^2+\frac{1}{3}T_4y^3$$

$$L_4=N_1x^4+N_2x^3y+N_3x^2y^2+N_4xy^3+N_5y^4 \qquad (4.2.9)$$

$$H_2=\frac{1}{2}\left(P_x^2+P_y^2+P_z^2\right)+n\left(yP_x-xP_y\right)+Ex^2+Gxy+Fy^2$$

$$=\frac{1}{2}\left(\dot{x}^2+\dot{y}^2\right)-\frac{n^2}{2}\left(x^2+y^2\right)+Ex^2+Gxy+Fy^2$$

$$H_3=-L_3$$

$$H_4=-L_4 \qquad (4.2.10)$$

where

$$E=-\frac{(1+\gamma)}{4}q_1f_{1xx}(0,0)-\frac{(1-\gamma)}{4}q_2f_{2xx}(0,0)$$

$$-\frac{A_1q_1(1+\gamma)}{8}f_{3xx}(0,0)-\frac{A_2q_2}{8}(1-\gamma)f_{4xx}(0,0)$$

 DOI : 10.35291/2454-9150.2026.0037

ISSN : 2454-9150 Vol-12, Issue-04, July 2026

$$F=-\frac{(1+\gamma)}{4}q_1 f_{1yy}(0,0)-\frac{(1-\gamma)}{4}q_2 f_{2yy}(0,0)$$

$$-\frac{A_1q_1}{8}(1+\gamma)f_{3yy}(0,0)-\frac{A_2q_2(1+\gamma)}{8}f_{4yy}(0,0)$$

$$G=-\frac{(1+\gamma)}{2}q_1 f_{1xy}(0,0)-\frac{(1-\gamma)}{2}q_2 f_{2xy}(0,0)$$

$$-\frac{A_1q_1}{4}(1+\gamma)f_{3xy}(0,0)-\frac{A_2q_2(1+\gamma)}{4}f_{4xy}(0,0)$$

$$T_1=\frac{(1-\mu)}{2}q_1 f_{1xxx}(0,0)+\frac{\mu q_2}{2}f_{2xxx}(0,0)$$

$$+\frac{A_1q_1(1-\mu)}{4}f_{3xxx}(0,0)+\frac{A_2q_2\mu}{4}f_{4xxx}(0,0)$$

$$T_2=(1-\mu)q_1 f_{1xxy}(0,0)+\mu q_2 f_{2xxy}(0,0)$$

$$+\frac{1}{2}A_1q_1(1-\mu)f_{3xxy}(0,0)+\frac{A_2q_2\mu}{2}f_{4xxy}(0,0)$$

$$T_3=\frac{(1-\mu)}{2}q_1 f_{1xyy}(0,0)+\frac{\mu q_2}{2}f_{2xyy}(0,0)$$

$$+\frac{A_1q_1}{4}(1-\mu)f_{3xyy}(0,0)+\frac{A_2q_2\mu}{4}f_{4xyy}(0,0)$$

$$T_4=(1-\mu)q_1 f_{1yyy}(0,0)+\frac{\mu}{2}q_2 f_{2yyy}(0,0)$$

$$+\frac{A_1q_1(1-\mu)}{4}f_{3yyy}(0,0)+\frac{A_2q_2\mu}{4}f_{4yyy}(0,0) \tag{4.2.11}$$

Now

$$f_{1xx}(0,0)=-\frac{1}{4}+\frac{3}{2}A_1-\frac{9}{8}A_2-\frac{3}{4}(1-q_1)+(1-q_2)+\frac{11}{8}A_1(1-q_1)-\frac{5}{3}A_2(1-q_2)+4A_2(1-q_2)$$

$$f_{2xx}(0,0)=-\frac{1}{4}-\frac{9}{8}A_1+\frac{3}{2}A_2+(1-q_1)-\frac{3}{4}(1-q_2)-\frac{5}{3}A_1(1-q_2)+\frac{11}{8}A_2(1-q_2)-\frac{5}{3}A_1(1-q_2)+4A_1(1-q_1)$$

$$f_{3xx}(0,0)=3\Big[\frac{1}{4}+\frac{5}{2}A_1-\frac{5}{8}A_2-\frac{5}{12}(1-q_1)+\frac{5}{3}(1-q_2) \\ +\tfrac{125}{24}A_1(1-q_1)-\tfrac{7}{5}A_2(1-q_2)-\tfrac{35}{6}A_2(1-q_2)\Big]$$

$$f_{4xx}(0,0)=3\left[1+\frac{7}{2}A_1+\frac{7}{3}(1-q_2)+\frac{7}{2}A_2(1-q_2)\right]$$

DOI : 10.35291/2454-9150.2026.0037

$$\left[1-2A_1+2A_2+\frac{3}{4}+\frac{A_1+A_2}{2}+\frac{1-q_1}{3}+\frac{1-q_2}{3}\right.$$

$$\left.+\frac{A_1}{2}(1-q_1)+\frac{A_2}{2}(1-q_2)\right]-\frac{(1+\gamma)}{4}q_1 f_{1xx}(0,0)$$

$$=\frac{1}{16}\left[q_1+q_2-6(A_1q_1+A_2q_2)+\frac{9}{2}(A_2q_1+A_1q_2)\right.$$

$$\left.+\gamma\left\{q_1-q_2+6(A_2q_2-A_1q_1)+\frac{9}{2}(A_2q_1-A_1q_2)\right\}\right]$$

$$-\frac{A_1q_1(1+\gamma)}{8}f_{3xx}(0,0)=-\frac{3}{32}(1+\gamma)A_1q_1$$

$$-\frac{A_2q_2(1-\gamma)}{8}f_{4xx}(0,0)=-\frac{3}{32}A_2q_2(1-\gamma)$$

$$-\frac{A_1q_1}{8}(1+\gamma)3(a^2+b^2)^{-\frac{7}{2}}(4a^2-b^2)$$

$$-\frac{A_2q_2}{8}(1-\gamma)\left[3\{(a+1)^2+b^2\}^{-\frac{7}{2}}\right][4(a+1)^2+b^2]$$

$$=-\frac{3}{32}[A_1q_1+A_2q_2+\gamma(A_1q_1-A_2q_2)]$$

$$q_1f_{1yy}(0,0)=\frac{5}{4}q_1+\frac{21}{8}A_2q_1-\frac{3}{2}A_1q_1+\frac{7}{4}q_1(1-q_1)-q_1(1-q_2)$$

$$q_2f_{2yy}(0,0)=\frac{5}{4}q_2+\frac{21}{8}A_1q_2-\frac{3}{2}A_2q_2+\frac{7}{4}q_2(1-q_2)-q_2(1-q_2)\mu$$

$$A_1q_1f_{3yy}(0,0)=\frac{33}{4}A_1q_1$$

$$A_2q_2f_{4yy}(0,0)=\frac{33}{4}A_2q_2$$

$$q_1f_{1xy}(0,0)=-\frac{3\sqrt{3}}{4}\left[q_1+\frac{7}{6}A_2q_1+\frac{2}{3}A_1q_1+\frac{7}{9}q_1(1-q_1)+\frac{4}{9}q_1(1-q_2)\right]$$

$$q_2f_{2xy}(0,0)=\frac{3\sqrt{3}}{4}\left[q_2+\frac{7}{6}A_1q_2+\frac{2}{3}A_2q_2+\frac{4}{9}q_2(1-q_1)+\frac{7}{9}q_2(1-q_2)\right]$$

$$A_1q_1f_{3yy}(0,0)=-\frac{15\sqrt{3}}{4}A_1q_1$$

$$A_2q_2f_{4yy}(0,0)=\frac{15\sqrt{3}}{4}A_2q_2$$

$$f_{1xxx}(0,0)=-\frac{21}{8}+\frac{9}{8}A_1+\frac{11}{8}A_2-\frac{31}{8}(1-q_1)+\frac{3}{4}(1-q_2)$$
$$+\frac{1}{2}A_1(1-q_1)+\frac{9}{8}A_2(1-q_2)+\frac{45}{8}A_2(1-q_1)$$

$$q_1f_{1xxx}(0,0)=-\frac{21}{8}q_1+\frac{9}{8}A_1q_1+\frac{11}{8}q_1A_2$$

$$\frac{1-\mu}{2}q_1f_{1xxx}(0,0)=\left(\frac{1+\gamma}{32}\right)[-21q_1+9A_1q_1+11q_1A_2]$$

$$f_{2xxx}(0,0)=\frac{2}{9}\left[1-A_1+A_2+\frac{2}{3}(1-q_1)-\frac{2}{3}(1-q_2)+A_1(1-q_1)\right.$$

$$-A_2(1-q_2)+\frac{5}{2}A_1+\frac{5}{3}(1-q_2)+\frac{5}{2}A_2(1-q_2)$$

 DOI : 10.35291/2454-9150.2026.0037

$$-\frac{15}{8}\Big[1-3A_1+3A_2+2(1-q_1)-2(1-q_2)$$

$$+3A_1(1-q_1)-3A_2(1-q_2)+\frac{7}{2}A_1+\frac{7}{3}(1-q_2)-\frac{7}{2}A_2(1-q_2)\Big]$$

$$q_2f_{2xxx}(0,0)=\frac{21}{8}q_2+\frac{93}{16}A_1q_2-\frac{9}{8}A_2q_2$$

$$\mu q_2f_{2xxx}(0,0)=\frac{(1-\gamma)}{32}\Big[21q_1+\frac{93}{2}A_1q_2-9A_2q_2\Big]$$

$$f_{3xxx}(0,0)=-\frac{45}{2}\Big[1+A_1-A_2-\frac{2}{3}(1-q_1)+\frac{2}{3}(1-q_2)-A_1(1-q_1)$$

$$+A_2(1-q_2)+\frac{7}{3}(1-q_1)+\frac{7}{3}A_1(1-q_1)+\frac{7}{2}A_2\Big]+\frac{105}{8}\Big[1+3A_1$$

$$-3A_2-2(1-q_1)+2(1-q_2)-3A_1(1-q_1)+3A_2(1-q_2)$$

$$+3(1-q_1)+\frac{9}{2}A_1(1-q_1)+\frac{9}{2}A_2\Big]$$

$$A_1q_1f_{3xxx}(0,0)=-\frac{75}{8}A_1q_1$$

$$A_1q_1\frac{(1-\mu)}{4}f_{3xxx}(0,0)=-\frac{75}{64}(1+\gamma)A_1q_1$$

$$f_{4xxx}(0,0)=\frac{45}{2}\Big[1-A_1+A_2+\frac{2}{3}(1-q_1)-\frac{2}{3}(1-q_2)+A_1(1-q_1)$$

$$-A_2(1-q_2)\Big]\Big[1+\frac{7}{2}A_1+\frac{7}{3}(1-q_2)+\frac{7}{2}A_2(1-q_2)\Big]$$

$$-\frac{105}{8}\Big[1-3A_1+3A_2+2(1-q_1)-2(1-q_2)$$

$$+3A_1(1-q_1)-3A_2(1-q_2)\Big]$$

$$A_2q_2f_{4xxx}(0,0)=\frac{45}{2}A_2q_2-\frac{105}{8}A_2q_2=\frac{75}{8}A_2q_2$$

$$\frac{\mu}{4}A_2f_{4xxx}(0,0)=\frac{75}{64}(1-\gamma)A_2q_2$$

$$f_{3xyy}(0,0)=-\frac{15}{2}\Big[1+A_1+\frac{5}{2}A_2+\frac{5}{3}(1-q_1)+\frac{2}{3}(1-q_2)+\frac{5}{2}A_1(1-q_1)$$

$$+A_2(1-q_2)\Big]+\frac{315}{8}\Big[1+\frac{A_1}{3}+\frac{17}{6}A_2-\frac{(1-q_1)}{9}+\frac{2}{9}(1-q_2)$$

$$+\frac{17}{6}A_1(1-q_1)+\frac{1}{3}A_2(1-q_2)\Big]$$

$$A_2q_2f_{4xyy}(0,0)=-\frac{255}{8}A_2q_2$$

 DOI : 10.35291/2454-9150.2026.0037

$$\frac{\mu}{4}A_2 q_2 f_{4xyy}(0,0) = -\frac{255}{8}A_2 q_2(1-\gamma),$$

$$f_{2xxy}(0,0) = -\frac{15\sqrt{3}}{2}\left[1+\frac{7}{6}A_1+\frac{5}{3}A_2+\frac{10}{9}(1-q_1)+\frac{7}{9}(1-q_2)\right.$$

$$\left.+\frac{5}{3}A_1(1-q_1)+\frac{7}{6}A_2(1-q_2)\right]+\frac{3\sqrt{3}}{2}\left[1+\frac{13}{6}A_1-\frac{1}{3}A_2\right.$$

$$\left.-\frac{2}{9}(1-q_1)+\frac{13}{9}(1-q_2)-\frac{1}{3}A_1(1-q_1)+\frac{13}{6}A_2(1-q_2)\right]$$

$$q_2 f_{2xxy}(0,0) = -\frac{3\sqrt{3}}{8}\left[q_2-\frac{17}{6}A_1q_2+\frac{29}{3}A_2q_2\right]$$

$$\mu q_2 f_{2xxy}(0,0) = -\frac{3\sqrt{3}}{8}(1-\gamma)\left[q_2-\frac{17}{6}A_1q_2+\frac{29}{3}A_2q_2\right]$$

$$A_1q_1(1-\mu)f_{3xxy}(0,0) = -\frac{45}{16}\sqrt{3}(1+\gamma)A_1q_1$$

$$\mu A_2q_2 f_{4xxy}(0,0) = -\frac{45}{16}\sqrt{3}(1-\mu)A_2q_2$$

$$(1-\mu)q_1 f_{1xxy}(0,0) = -\frac{3\sqrt{3}}{4}(1+\gamma)\left[\frac{q_1}{4}+\frac{29}{12}A_1q_1-\frac{17}{24}A_2q_1\right]$$

$$q_1 f_{1xyy}(0,0) = \frac{33}{8}q_1+\frac{3}{8}A_1q_1+\frac{129}{16}A_2q_1$$

$$(1-\mu)q_1 f_{1xyy}(0,0) = \left(\frac{1+\gamma}{2}\right)\left(\frac{33}{8}q_1+\frac{3}{8}A_1q_1+\frac{129}{16}A_2q_1\right)$$

$$q_2 f_{2xyy}(0,0) = -\frac{33}{8}q_2-\frac{129}{16}A_1q_2-\frac{3}{8}A_2q_2$$

$$\mu q_2 f_{2xyy}(0,0) = \left(\frac{1-\gamma}{2}\right)\left[-\frac{33}{8}q_2-\frac{129}{16}A_1q_2-\frac{3}{8}A_2q_2\right]$$

$$q_1 f_{1yyy}(0,0) = -\frac{\sqrt{3}}{8}\left[9q_1-33A_1q_1-\frac{151}{2}A_2q_1\right]$$

$$q_2 f_{2yyy}(0,0) = -\frac{\sqrt{3}}{8}\left[9q_2-33A_2q_2+\frac{69}{2}A_1q_2\right]$$

$$A_1q_1 f_{3yyy}(0,0) = -\frac{135}{8}\sqrt{3}A_1q_1$$

DOI : 10.35291/2454-9150.2026.0037

$$A_2 q_2 f_{4yyy}(0,0) = -\frac{135}{8}\sqrt{3}A_2 q_2 \tag{4.2.12}$$

Using these in (4.2.11), we have

$$E = \frac{1}{16}\Big[q_1 + q_2 - \frac{15}{2}(A_1q_1 + A_2q_2) + \frac{9}{2}(A_2q_1 + A_1q_2)$$
$$+\left\{q_1 - q_2 + \frac{15}{2}(A_2q_2 - A_1q_1) + \frac{9}{2}(A_2q_1 - A_1q_2)\right\}\Big]$$

$$F = -\frac{1}{16}\Big[5(q_1+q_2) - \frac{21}{2}(A_1q_1 + A_2q_2) + \frac{21}{2}(A_2q_1 + A_1q_2)$$
$$+7q_1(1-q_1) + 7q_2(1-q_2) - 4q_1(1-q_2) - 4q_2(1-q_1)$$
$$+\{5(q_1-q_2) + \frac{21}{2}(A_2q_1 - A_1q_2) + \frac{21}{2}(A_1q_1 - A_2q_2)$$
$$+7q_1(1-q_1) - 7q_2(1-q_2) - 4q_1(1-q_2) + 4q_2(1-q_1)\}\gamma\Big]$$

$$G = \frac{\sqrt{3}}{2}\Big[3(q_1-q_2) + \frac{7}{2}(A_2q_1 + A_1q_2) + \frac{19}{2}(A_1q_1 - A_2q_2)$$
$$+\frac{7}{3}(q_1-q_2)(1-q_2) + \frac{4}{3}\{q_1(1-q_2) - q_2(1-q_1)\}$$
$$+\gamma\Big\{3(q_1+q_2) + \frac{7}{2}(A_1q_2 + A_2q_1) + \frac{19}{2}(A_1q_1 + A_2q_2)$$
$$+\frac{4}{3}q_1(1-q_1) + \frac{4}{3}q_2(1-q_2) + \frac{7}{3}(q_1+q_2)(1-q_2)\Big\}\Big]$$

$$T_1 = \frac{1}{32}\Big[21(q_2-q_1) - \frac{57}{2}A_1q_1 + \frac{57}{2}A_2q_2 + 11q_1A_2 + \frac{93}{2}A_1q_2$$
$$-21(q_1+q_2)\gamma + \left\{-\frac{57}{2}A_1q_1 - \frac{93}{2}A_1q_2 + 11q_1A_2 - \frac{57}{2}A_2q_2\right\}\gamma\Big]$$

$$T_2 = -\frac{\sqrt{3}}{16}\Big[3(q_1+q_2) + \frac{103}{2}(A_1q_1 + A_2q_2) - \frac{17}{2}(A_1q_2 + A_2q_1)$$
$$+\left\{3(q_1-q_2) + \frac{103}{2}(A_1q_1 - A_2q_2) + \frac{17}{2}(A_1q_2 - A_2q_1)\right\}\gamma\Big]$$

$$T_3 = \frac{1}{4}\Big[\frac{33}{8}(q_1-q_2) + \frac{261}{16}A_1q_1 - \frac{255}{16}A_2q_2 - \frac{129}{16}A_1q_2 + \frac{123}{16}A_2q_1$$

 DOI : 10.35291/2454-9150.2026.0037

ISSN : 2454-9150 Vol-12, Issue-04, July 2026

$$+\left\{\frac{33}{8}(q_1+q_2)+\frac{261}{16}A_1q_1+\frac{261}{16}A_2q_2+\frac{129}{16}A_1q_2+\frac{129}{16}A_2q_1\right\}\gamma\Bigg]$$

$$T_4=-\frac{\sqrt{3}}{32}\Bigg[9(q_1+q_2)-33(A_1q_1-A_2q_2)-\frac{151}{2}A_2q_1+\frac{69}{2}A_1q_2$$

$$+\frac{135}{2}(A_1q_1+A_2q_2)+\gamma\left\{9(q_1-q_2)+\frac{69}{2}A_1q_1-\frac{69}{2}A_1q_2-\frac{151}{2}A_2q_1-\frac{69}{2}A_2q_2\right\}\Bigg]$$

and

$$N_1=\frac{(1-\mu)}{24}q_1f_{1xxxx}(0,0)+\frac{\mu q_2}{24}f_{2xxxx}(0,0)+\frac{A_1q_1(1-\mu)}{48}f_{3xxxx}(0,0)+\frac{A_2q_2\mu}{48}f_{4xxxx}(0,0)$$

$$N_2=\frac{(1-\mu)q_1}{6}f_{1xxxy}(0,0)+\frac{\mu q_2}{6}f_{2xxxy}(0,0)+\frac{A_1q_1(1-\mu)}{12}f_{3xxxy}(0,0)+\frac{A_2q_2\mu}{12}f_{4xxxy}(0,0)$$

$$N_3=\frac{(1-\mu)q_1}{4}f_{1xxyy}(0,0)+\frac{\mu q_2}{4}f_{2xxyy}(0,0)+\frac{A_1q_1(1-\mu)}{8}f_{3xxyy}(0,0)+\frac{A_2q_2\mu}{8}f_{4xxyy}(0,0)$$

$$N_4=\frac{(1-\mu)q_1}{6}f_{1xyyy}(0,0)+\frac{\mu q_2}{6}f_{2xyyy}(0,0)+\frac{A_1q_1(1-\mu)}{12}f_{3xyyy}(0,0)+\frac{A_2q_2\mu}{12}f_{4xyyy}(0,0)$$

$$N_5=\frac{(1-\mu)q_1}{24}f_{1yyyy}(0,0)+\mu q_2f_{2yyyy}(0,0)+\frac{A_1q_1(1-\mu)}{48}f_{3yyyy}(0,0)+\frac{A_2q_2\mu}{48}f_{4yyyy}(0,0)$$

with

$$f_{1xxxx}(0,0)=-\frac{111}{16}-\frac{75}{4}A_1+\frac{255}{32}A_2-\frac{85}{16}(1-q_1)-\frac{25}{2}(1-q_2)$$

$$f_{1xxxy}(0,0)=f_{1yxxx}(0,0)=\sqrt{3}\left[\frac{75}{16}-10A_1+\frac{535}{32}A_2+\frac{2015}{96}(1-q_1)-\frac{20}{3}(1-q_2)\right]$$

$$f_{1xxyy}(0,0)=-\frac{123}{16}+\frac{105}{4}A_1+\frac{195}{32}A_2+\frac{65}{16}(1-q_2)-\frac{35}{2}(1-q_2)$$

$$f_{1xyyy}(0,0)=\sqrt{3}\left[-\frac{135}{16}+\frac{15}{2}A_1-\frac{795}{32}A_2-\frac{265}{16}(1-q_1)+5(1-q_2)\right]$$

$$f_{1yyyy}(0,0)=-\frac{9}{16}-\frac{135}{4}A_1+\frac{585}{32}A_2+\frac{155}{16}(1-q_1)-\frac{45}{2}(1-q_2)$$

$$f_{3xxxx}(0,0)=-\frac{855}{16}-\frac{315}{4}A_1-\frac{1725}{32}A_2-\frac{1575}{16}(1-q_1)-\frac{105}{2}(1-q_2)$$

DOI : 10.35291/2454-9150.2026.0037

$$f_{3xxxy}(0,0)=\sqrt{3}\left[-\frac{315}{16}-105A_1+\frac{3885}{32}A_2+\frac{6615}{16}(1-q_1)-\frac{70}{3}(1-q_2)\right]$$

$$f_{3xyyy}(0,0)=\sqrt{3}\left[-\frac{1575}{16}+\frac{105}{2}A_1-\frac{11865}{32}A_2+\frac{1365}{16}(1-q_1)+35(1-q_2)\right]$$

$$f_{3xxyy}(0,0)=\frac{1395}{16}+\frac{945}{4}A_1+\frac{5985}{32}(1-q_1)-\frac{315}{2}(1-q_2)$$

$$f_{3yyyy}(0,0)=-\frac{855}{16}-\frac{2835}{4}A_1-\frac{14805}{32}A_2-\frac{4935}{16}(1-q_1)-\frac{945}{2}(1-q_2)$$

(4.2.13)

III. PERTURBED BASIC FREQUENCIES

Now we follow the method of Whittakar (1965) to find the canonical transformation from the phase space (x, y, p_x, p_y) in to the phase space product of the angle co-ordinates (ϕ_1,ϕ_2) and the action momenta (I_1, I_2) and of the first order in $I_1^{1/2}, I_2^{1/2}$ so that the second order part of the Hamiltonian is normalized.

In order to examine the stability of the motion, we consider the following set of linear equations in the variable x,y :

$$-\lambda P_x=\frac{\partial H_2}{\partial x}=2Ex+Gy-nP_y$$

$$-\lambda P_y=\frac{\partial H_2}{\partial y}=2Ey+Gx+nP_x \qquad (4.3.1)$$

$$\lambda x=\frac{\partial H_2}{\partial P_x}=P_x+ny$$

$$\lambda y=\frac{\partial H_2}{\partial P_y}=P_y-nx$$

i.e $AX=0$

where

$$X=\begin{bmatrix} x\\ y\\ P_x\\ P_y\end{bmatrix} \quad \text{and} \quad A=\begin{bmatrix} 2E & G & \lambda & -n\\ G & 2F & n & \lambda\\ \lambda & n & 1 & 0\\ -n & -\lambda & 0 & 1\end{bmatrix}$$

Clearly, $|A|=0$ implies that

$$\lambda^4+2(E+F+n^2)\lambda^2+4EF-G^2+n^4-2n^2(E+F)=0 \qquad (4.3.2)$$

This is the characteristic equation whose discriminant is

$$D=4(E+F+n^2)^2-4\{4EF-G^2+n^4-2n^2(E+F)\} \qquad (4.3.3)$$

Stability is possible only when D > 0,

 DOI : 10.35291/2454-9150.2026.0037

This implies that [Singh and Ishwar (1999)]

$$\mu < \mu_{co} = \mu_0 - \frac{2}{9\sqrt{69}}(A_1 + A_2) - \frac{2(1-q_1)}{27\sqrt{69}}$$
$$-\frac{1}{9}(1 + \frac{11}{\sqrt{69}}A_1 q_1 - \frac{2(1-q_2)}{27\sqrt{69}} + \frac{1}{9}(1 - \frac{11}{\sqrt{69}}A_2 q_2$$

where

$$\mu_0 = \frac{1}{2}\left(1 - \sqrt{\frac{23}{27}}\right) = 0.0385208965 \tag{4.3.4}$$

The roots of the equation (4.3.2) are denoted by $\pm\, i\omega_1$ and $\pm\, i\omega_2$ where ω_1, ω_2 are the long /short periodic perturbed basic frequencies.

These roots are related to each other as

$$\omega_1^2 + \omega_2^2 = 2\left(E + F + n^2\right)$$
$$= \frac{1}{16}\left[-8(q_1 + q_2) - 36(A_1 q_1 + A_2 q_2) - 12(A_2 q_1 + A_1 q_2)\right.$$
$$-14q_1(1-q_1) - 14q_2(1-q_2) + 8q_1(1-q_2) + 8q_2(1-q_1)$$
$$+32 + 48(A_1 + A_2) + \gamma\{-8q_1 + 8q_2 + 36(A_2 q_2 - A_1 q_1)$$
$$+12(A_1 q_2 - A_2 q_1) - 14q_1(1-q_1) + 14q_2(1-q_2)$$
$$\left.+8q_1(1-q_2) - 8q_2(1-q_1)\}\right] \tag{4.3.5}$$

and

$$0 < \omega_2 < \frac{1}{\sqrt{2}} < \omega_1 < 1$$

$$\omega_1^2\omega_2^2 = 4EF - G^2 + n^4 - 2n^2(E+F)$$
$$= \frac{1}{64}\left[-5(q_1 + q_2)^2 + 27(q_1 + q_2)(A_1 q_1 + A_2 q_2)\right.$$
$$-33(q_1 + q_2)(A_2 q_1 + A_1 q_2) + 64 + 192(A_1 + A_2) + 32(q_1 + q_2)$$
$$+192(A_1 q_1 + A_2 q_2) + 96(A_1 q_2 + A_2 q_1) + 56q_1(1-q_1) + 56q_2(1-q_2)$$
$$-32q_1(1-q_2) - 32q_2(1-q_1) + \gamma\{32q_1 - 32q_2 + 192(A_1 q_1 - A_2 q_2)$$
$$+96(A_2 q_1 - A_1 q_2) - 10(q_1^2 - q_2^2) - 33(q_1 + q_2)(A_2 q_1 - A_1 q_2)$$
$$-27(q_1 + q_2)(A_2 q_2 - A_1 q_1) - 54(q_1^2 - q_2^2) - 63(q_1 - q_2)(A_1 q_2 + A_2 q_1)$$
$$-171(q_1 - q_2)(A_1 q_1 + A_2 q_2) - 63(q_1 + q_2)(A_2 q_1 - A_1 q_2)$$
$$-171(q_1 + q_2)(A_1 q_1 - A_2 q_2)\} - \gamma^2\{27(q_1 + q_2)^2$$
$$\left.+63(q_1 + q_2)(A_1 q_2 + A_2 q_1) + 171(q_1 + q_2)(A_1 q_1 - A_2 q_2)\}\right] \tag{4.3.6}$$

IV. NORMAL FORM OF SECOND ORDER PART OF HAMILTONIAN

In order to determine the normal coordinates, we first solve the set of equations (4.3.1). Its solutions are:

 DOI : 10.35291/2454-9150.2026.0037

$$\frac{x}{2n\lambda - G} = \frac{y}{\lambda^2 - n^2 + 2E} = \frac{P_x}{n\lambda^2 - G\lambda - 2nE + n^3} = \frac{P_y}{\lambda^3 + n^2\lambda + 2E\lambda - nG}$$

(4.4.1)

Putting $\lambda = \pm i\omega_1$ and $\lambda = \pm i\omega_2$, we get the following set of solutions

$$x_j = K_j(2ni\omega_j - G)$$

$$P_{x_j} = K_j(-n\omega_j^2 - iG\omega_j - 2En + n^3)$$

$$y_j = K_j(-\omega_j^2 - n^2 + 2E)$$

$$P_{y_j} = K_j[-i\omega_j^3 + i\omega_j(n^2 + 2E) - Gn] \qquad (4.4.2)$$

$$x_{j+2} = K_{j+2}(-2in\omega_j - G)$$

$$P_{x_{j+2}} = K_{j+2}(-n\omega_j^2 + iG\omega_j - 2En + n^3)$$

$$y_{j+2} = K_{j+2}(-\omega_j^2 - n^2 + 2E)$$

$$P_{y_{j+2}} = K_{j+2}\{i\omega_j^3 - i\omega_j(n^2 + 2E) - Gn\}$$

where j = 1,2 and K_j and K_{j+2} are constants of proportionality.

As in Whittaker (1965), we use the transformation

$$X = J\begin{pmatrix} Q_1 \\ Q_2 \\ P_1 \\ P_2 \end{pmatrix} \qquad (4.4.3)$$

with

$$J = \begin{pmatrix} x_1 - \frac{i\omega_1 x_3}{2} & -x_2 - \frac{i\omega_2 x_4}{2} & -i\frac{x_1}{\omega_1} + \frac{x_3}{2} & -i\frac{x_2}{\omega_2} - \frac{x_4}{2} \\ y_1 - \frac{i\omega_1 y_3}{2} & -y_2 - \frac{i\omega_2 y_4}{2} & -i\frac{y_1}{\omega_1} + \frac{y_3}{2} & -i\frac{y_2}{\omega_2} - \frac{y_4}{2} \\ P_{x,1} - \frac{i\omega_1 P_{x,3}}{2} & -P_{x,2} - \frac{i\omega_2 P_{x,4}}{2} & -i\frac{P_{x,1}}{\omega_1} + \frac{P_{x,3}}{2} & -i\frac{P_{x,2}}{\omega_2} - \frac{P_{x,4}}{2} \\ P_{y,1} - \frac{i\omega_1 P_{y,3}}{2} & -P_{y,2} - \frac{i\omega_2 P_{y,4}}{2} & -i\frac{P_{y,1}}{\omega_1} + \frac{P_{y,3}}{2} & -i\frac{P_{y,2}}{\omega_2} - \frac{P_{y,4}}{2} \end{pmatrix} \qquad (4.4.4)$$

Under the normality conditions, we have

$$x_1 p_{x,3} - x_3 p_{x,1} + y_1 p_{y,3} - y_3 p_{y,1} = 1$$

$$x_2 p_{x,4} - x_4 p_{x,2} + y_2 p_{y,4} - y_4 p_{y,2} = 1$$

Which give

$$-4i\omega_1 K_1 K_3[\omega_1^2(F - E + 2n^2) + G^2 + 2E^2 + 3n^2E + n^2F - 2EF - 2n^4] = 1$$

$$-4i\omega_2 K_2 K_4[\omega_2^2(F - E + 2n^2) + G^2 + 2E^2 + 3n^2E + n^2F - 2EF - 2n^4] = 1 \qquad (4.4.5)$$

Where K_j are arbitrary,

DOI : 10.35291/2454-9150.2026.0037

Here K_j (j = 1,2,3,4) satisfy the two equations (4.4.5). They can be made to satisfy two more equations. For this, we follow the method of Breakwell and Prngle (1966). We choose K_j so as to satisfy the equations

$$J_{1,1} = J_{1,2} = 0$$

This implies that

$$K_1(2in\omega_1 - G) = \frac{\omega_1 K_3}{2}(2n\omega_1 - iG)$$

$$K_2(G - 2in\omega_2) = \frac{\omega_2 K_4}{2}(-iG + 2n\omega_2) \tag{4.4.6}$$

From equations (4.4.6), we have

$$\frac{K_1}{\omega_1(2n\omega_1 - iG)} = \frac{K_3}{2(2in\omega_1 - G)} = h_1, \tag{4.4.7}$$

$$\frac{K_2}{\omega_2(2n\omega_2 - iG)} = \frac{K_4}{2(G - 2in\omega_2)} = h_2 . \tag{4.4.8}$$

Using (4.4.7) and (4.4.8) in (4.4.5), we have

$$h_j = \frac{1}{2\omega_j M_j \overline{M}_j (M_j^*)^2} \tag{4.4.9}$$

where

$$M_j = (\omega_j^2 - 2F + n^2)^{\frac{1}{2}}$$

$$= \frac{1}{2}[I_j + 6(A_1 + A_2) + \frac{5}{2}(q_1 + q_2) - 5 + \frac{21}{4}(A_1 q_1 - A_2 q_2) +$$

$$+ \frac{21}{4}(A_2 q_1 + A_1 q_2) + \frac{7}{2}q_1(1 - q_1) + \frac{7}{2}q_2(1 - q_2) - 2q_1(1 - q_2)$$

$$- 2q_2(1 - q_2) + \gamma\left\{\frac{5}{2}(q_1 - q_2) + \frac{21}{4}(A_2 q_1 - A_1 q_2) + \frac{21}{4}(A_1 q_1 - A_2 q_2)\right.$$

$$\left. + \frac{7}{2}q_1(1 - q_1) - \frac{7}{2}q_2(1 - q_2) - 2q_1(1 - q_2) + 2q_2(1 - q_1)\right\}]^{\frac{1}{2}}$$

with $I_j^2 = 4\omega_j^2 + 9, \quad j = 1,2$.

$$\overline{M}_1 = \sqrt{2}(\omega_1^2 - E - F + n^2)^{\frac{1}{2}}$$

$$= [K_1^2 - \frac{1}{16}\{8(q_1 + q_2) - 36(A_1 q_1 + A_2 q_2) - 12(A_2 q_1 + A_1 q_2)$$

$$- 14q_1(1 - q_1) - 14q_2(1 - q_2) + 8q_1(1 - q_2) + 8q_2(1 - q_1)$$

 DOI : 10.35291/2454-9150.2026.0037

$$+16+48(A_1+A_2)\}-\frac{1}{16}\gamma\{-8q_1+8q_2+36(A_2q_2-A_1q_1)$$

$$+12(A_1q_2-A_2q_1)-14q_1(1-q_1)+14q_2(1-q_2)$$

$$+8q_1(1-q_2)-8q_2(1-q_1)\}]^{\frac{1}{2}} \text{ with } K_1^2=2\omega_1^2-1,$$

$$\overline{M}_2=\sqrt{2}\left(\omega_2^2-E-F+n^2\right)^{\frac{1}{2}}=i\left[-2\omega_2^2+2E+2F+2n^2\right]^{\frac{1}{2}}$$

$$=i[K_2^2+\frac{1}{16}\{-8(q_1+q_2)-36(A_1q_1+A_2q_2)-12(A_2q_1+A_1q_2)$$

$$-14q_1(1-q_1)-14q_2(1-q_2)+8q_1(1-q_2)+8q_2(1-q_1)$$

$$+16+48(A_1+A_2)\}-\frac{\gamma}{16}\{-8q_1+8q_2+36(A_2q_2-A_1q_1)$$

$$+12(A_1q_2-A_2q_1)-14q_1(1-q_1)+14q_2(1-q_2)$$

$$+8q_1(1-q_2)-8q_2(1-q_1)\}]^{\frac{1}{2}} \text{ with } K_2^2=-(2\omega_2^2-1).$$

$$M_j^*=\left(\omega_j^2-2E+n^2\right)^{\frac{1}{2}}$$

$$=\frac{1}{2}[I_j^2+6(A_1+A_2)-\frac{1}{2}(q_1+q_2)-5+\frac{15}{4}(A_1q_1-A_2q_2)+$$

$$-\frac{9}{4}(A_2q_1+A_1q_2)-\frac{\gamma}{2}\left\{q_1-q_2+\frac{15}{2}(A_2q_2-A_1q_1)+\frac{9}{2}(A_2q_1-A_1q_2)\right\}\Bigg]^{\frac{1}{2}}$$

(4.4.10)

$$J=\begin{bmatrix} 0 & 0 & \dfrac{M_1}{\omega_1\overline{M}_1} & \dfrac{iM_2}{\omega_2\overline{M}_2} \\ -\dfrac{2n\omega_1}{M_1\overline{M}_1} & 2in\omega_2 & \dfrac{G}{\omega_1M_1\overline{M}_1} & \dfrac{iG}{\omega_2M_2\overline{M}_2} \\ -\dfrac{\omega_1\left(M_1^2-2n^2\right)}{M_1\overline{M}_1} & -\dfrac{i\omega_2\left(M_2^2-2n^2\right)}{M_2\overline{M}_2} & \dfrac{-nG}{\omega_1M_1\overline{M}_1} & \dfrac{-niG}{\omega_2M_2\overline{M}_2} \\ \dfrac{-\omega_1G}{M_1\overline{M}_1} & \dfrac{-\omega_2G}{M_2\overline{M}_2} & \dfrac{-n\left(2\omega_1^2-M_1^2\right)}{\omega_1M_1\overline{M}_1} & \dfrac{ni\left(M_2^2-2\omega_2^2\right)}{\omega_2M_2\overline{M}_2} \end{bmatrix}$$

(4.4.11)

where

$$J_{1,3}=\frac{M_1}{\omega_1\overline{M}_1}=\frac{I_1}{2\omega_1k_1}\left[1+\frac{1}{2I_1^2}a_1+\frac{\gamma}{32k_1^2}a_2+\frac{\gamma}{2I_1^2}a_3+\frac{\gamma}{32k_1^2}a_4\right],$$

 DOI : 10.35291/2454-9150.2026.0037

$$J_{1,4}=\frac{I_1}{2\omega_2 k_2}\left[1+\frac{1}{2I_2^2}a_1-\frac{1}{32k_2^2}a_2+\frac{\gamma}{2I_2^2}a_3-\frac{\gamma}{32k_2^2}a_4\right],$$

$$J_{2,1}=-\frac{4\omega_1}{I_1k_1}\left[1+a_5-\frac{1}{2I_1^2}a_6+\frac{1}{32k_1^2}a_7-\frac{\gamma}{2I_1^2}a_8+\frac{\gamma}{32k_1^2}a_9\right],$$

$$J_{2,2}=-\frac{4\omega_1}{I_2k_2}\left[1+a_5-\frac{1}{2I_2^2}a_6-\frac{1}{32k_2^2}a_7-\frac{\gamma}{2I_2^2}a_8-\frac{\gamma}{32k_2^2}a_9\right],$$

$$J_{2,3}=-\frac{\sqrt{3}}{4\omega_1 I_1 k_1}\left[a_{10}+a_{11}\gamma-\frac{1}{2I_1^2}a_{12}+\frac{1}{32k_1^2}a_{13}-\frac{1}{2I_1^2}a_{14}\gamma\right.$$
$$\left.-\frac{1}{2I_1^2}a_{15}\gamma^2+\frac{1}{32k_1^2}a_{16}\gamma+\frac{1}{32k_1^2}a_{17}\gamma\right]$$

$$J_{2,4}=-\frac{\sqrt{3}}{4\omega_2 I_2 k_2}\left[a_{10}-a_{11}\gamma-\frac{1}{2I_2^2}a_{12}-\frac{1}{32k_2^2}a_{13}-\frac{1}{2I_2^2}a_{14}\gamma\right.$$
$$\left.-\frac{1}{2I_2^2}a_{15}\gamma^2+\frac{1}{32k_2^2}a_{16}\gamma+\frac{1}{32k_2^2}a_{17}\gamma^2\right] \tag{4.4.12}$$

$$I_j^2=4\omega_j^2+9,$$
$$k_1^2=2\omega_1^2-1,$$
$$k_2^2=-(2\omega_2^2-1),\ j=1,2 \tag{4.4.13}$$

$$a_1'=6(A_1+A_2)+\frac{5}{2}(q_1+q_2)-5+\frac{21}{4}(A_1q_1+A_2q_2)+\frac{21}{4}(A_2q_1+A_1q_2),$$

$$a_2'=-8(q_1+q_2)-36(A_1q_1+A_2q_2)-12(A_2q_1+A_1q_2)+16+48(A_1+A_2),$$

$$a_3'=\frac{5}{2}(q_1-q_2)+\frac{21}{4}(A_2q_1-A_1q_2)+\frac{21}{4}(A_1q_1-A_2q_2),$$

$$a_4'=-8q_1+8q_2+36(A_2q_2-A_1q_1)+12(A_1q_2+A_2q_1),$$

$$a_5'=\frac{3}{4}(A_1+A_2),$$

$$a_6'=\frac{9}{4}(A_1+A_2)+\frac{5}{2}(q_1+q_2)-5+\frac{57}{2}(A_1q_1+A_2q_2+A_1q_1+A_2q_1),$$

$$a_7'=-8(q_1+q_2)+16+60(A_1+A_2)-42(A_1q_1+A_2q_2)-18(A_1q_2+A_2q_1),$$

 DOI : 10.35291/2454-9150.2026.0037

$$a_8' = \frac{5}{2}(q_1 - q_2) + \frac{57}{8}A_1q_1 - \frac{57}{8}A_2q_2 + \frac{57}{8}A_2q_1 - \frac{57}{8}A_1q_2,$$

$$a_9' = -8(q_1 - q_2) - 30(A_1q_1 - A_2q_1) + 6(A_1q_2 - A_2q_1),$$

$$a_{10}' = 3(q_1 - q_2) + \frac{7}{2}(A_2q_1 - A_1q_2) + \frac{19}{2}(A_1q_1 - A_2q_2),$$

$$a_{11}' = 3(q_1 + q_2) + \frac{7}{2}(A_1q_2 + A_2q_1) + \frac{19}{2}(A_1q_1 + A_2q_2),$$

$$a_{12}' = -\frac{59}{2}A_1q_1 - \frac{1}{2}A_1q_2 + \frac{1}{2}A_2q_1 + \frac{59}{2}A_2q_2 + \frac{15}{2}q_1^2 + \frac{49}{2}A_2q_1^2 + 15A_1q_1q_2$$

$$+\frac{79}{2}A_1q_1^2 - 15A_2q_1q_2 - \frac{15}{2}q_2^2 - \frac{49}{2}A_1q_2^2 - \frac{79}{2}A_2q_2^2 - 15q_1 + 15q_2,$$

$$a_{13}' = -24q_1^2 - 184A_1q_1^2 - 24A_2q_1q_2 - 64A_2q_1^2 + 64A_1q_1q_2 + 48q_1 + 296A_1a_1$$

$$+24q_2^2 + 184A_2q_2^2 - 296A_2q_2 + 64A_1q_2^2 - 48q_2 + 200A_2q_1 - 200A_1q_2,$$

$$a_{14}' = 15q_1^2 + 21A_2q_1^2 + \frac{63}{4}A_1q_1q_2 + 79A_1q_1^2 + \frac{63}{4}A_2q_1q_2 + 15q_2^2 + 21A_1q_2^2$$

$$+79A_1q_2^2 - \frac{59}{2}A_1q_1 + \frac{1}{2}A_2q_1 + \frac{1}{2}A_1q_2 - \frac{59}{2}A_2q_2 - 15q_1 - 15q_2,$$

$$a_{15}' = \frac{15}{2}q_1^2 - \frac{15}{2}q_2^2 + \frac{49}{2}A_2q_1^2 + \frac{79}{2}A_1q_1^2 - \frac{79}{2}A_2q_2^2$$

$$-\frac{49}{2}A_1q_2^2 - 15A_1q_1q_2 + 15A_2q_1q_2,$$

$$a_{16}' = -48q_1^2 - 48q_2^2 - 128A_2q_1^2 - 368A_1q_1^2 - 368A_2q_2^2 - 128A_1q_2^2 + 48q_1$$

$$+48q_2 + 200A_1q_2 + 296A_2q_2 + 200A_2q_1 + 296A_1q_1,$$

$$a_{17}' = -24q_1^2 + 24q_2^2 + 24A_2q_1q_2 - 184A_1q_1^2 - 24A_2q_1q_2 - 64A_2q_1^2$$

$$+184A_2q_2^2 + 64A_1q_2^2 \tag{4.4.14}$$

The transformation (4.4.3) transforms H_2 to the form:

$$H_2 = \frac{1}{2}\left(P_1^2 - P_2^2 + \omega_1^2Q_1^2 - \omega_2^2Q_2^2\right) \tag{4.4.15}$$

We apply a contact transformation Q_1, Q_2, P_1, P_2 to Q_1', Q_2', P_1', P_2' defined by Whittaker (1965):

$$Q_j = \sqrt{\frac{2Q_j'}{\omega_j}}\sin P_j',$$

DOI : 10.35291/2454-9150.2026.0037

$$P_j = \sqrt{2\omega_j Q_j'} \cos P_j', \qquad (j = 1,2)$$

$$H_2 = \omega_1 Q_1' - \omega_2 Q_2'$$

We denote the angular variables and then second order part of the Hamiltonian becomes P_1' and P_2' by ϕ_1 and ϕ_2 and, the actions Q_1' and Q_2' by I_1 and I_2. Then the second order part of the Hamiltonian is transformed to the normal form:

$$H_2 = \omega_1 I_1 - \omega_2 I_2 \qquad (4.4.16)$$

Taking $H = H_0 - H_2$, equations of motion

$$\frac{dI_j}{dt} = -\frac{\partial H}{\partial \phi_j},$$

$$\frac{d\phi_j}{dt} = -\frac{\partial H}{\partial I_j} \qquad (j = 1,2)$$

become

$$\frac{dI_j}{dt} = 0, \qquad (j = 1,2)$$

$$\frac{d\phi_1}{dt} = \omega_1,$$

$$\frac{d\phi_2}{dt} = -\omega_2 \qquad (4.4.17)$$

The general solutions of these equations of motion is

$$I_j = \text{Constant},$$

$$\phi_1 = \omega_1 t + \text{constant},$$

$$\phi_2 = -\omega_2 t + \text{constant} \qquad (4.4.18)$$

V. CONCLUSION

We conclude that the Hamiltonian is affected by the radiation repulsive forces and oblateness of the primaries. They also perturb the basic frequencies. The transformations utilized for reduction of Hamiltonian to the normal form are dependent on the perturbed basic frequencies.

REFERENCES.

1. ***Kumar, M., & Kumar, A. (2026).*** *Stability of triangular equilibrium points in the generalised photogravitational restricted three body problem. International Journal of Scientific Research in Science and Technology, 13(1), 198–207.* https://doi.org/10.32628/IJSRST26132
2. ***Saputra, M. B., et al. (2026).*** *Stability of equilibrium points in modified elliptic restricted three-body problem with various perturbation sources. Few-Body Systems, 67(1).*
3. ***Doucette, D. (2026).*** *The barrier power of L4/L5 and the gateway power of L1/L2. SAIT Polytechnic Research.*

 DOI : 10.35291/2454-9150.2026.0037

4. **Aliyu, S. A., & Ismail, H. A. (2026).** *Investigating the influence of radiation pressure on the stability of Lagrangian points in celestial mechanics. Journal of Multidisciplinary Science: MIKAILALSYS, 4(1), 41–59.*
5. **Litteri, W., et al. (2026).** *Generation of periodic orbits in the restricted three-body problem with a variational autoencoder. ResearchGate Publication.*
6. **Peterson, L. T. (2026).** *Lyapunov-based controller for low-thrust orbital transfers in the circular restricted three-body problem. Journal of Guidance, Control, and Dynamics.*
7. **Kumar, M., & Kumar, A. (2025).** *Existence of equilibrium points in the generalised photogravitational restricted three body problem space. International Journal of Scientific Research in Science and Technology, 12(5), 179–185. https://doi.org/10.32628/IJSRST2513214*
8. **Hunsberger, C., et al. (2025).** *Accounting for solar radiation pressure in the Hamiltonian normal form of the elliptic restricted three-body problem. arXiv:2512.03942.*
9. **Almanza-Soto, J. P., & Howell, K. C. (2025).** *Persistence of restricted three-body problem normal form structures in higher-fidelity models. Celestial Mechanics and Dynamical Astronomy, 137.*
10. **Richard, T. K. (2025).** *On the existence and stability of collinear equilibrium points in the photogravitational circular restricted three-body problem. FUDMA Journal of Sciences, 9(6), 202–207.*
11. **Singh, J., & Omale, S. (2019).** *Photogravitational restricted three-body problem with oblateness. Astrophysics and Space Science, 364(1), 1–12. https://doi.org/10.1007/s10509-019-3527-4*
12. **Mishra, V. K., & Ishwar, B. (2016).** *Hamiltonian normalization in the generalized photogravitational restricted three-body problem. International Journal of Applied Mechanics and Engineering, 21(4), 1007–1020.*
13. **Kushvah, B. S., Sharma, J. P., & Ishwar, B. (2006).** *Stability of Lagrangian points in the photogravitational restricted three-body problem. Astrophysics and Space Science, 305, 319–331. https://doi.org/10.1007/s10509-006-9119-2*
14. **Kalantonis, V. S., Perdios, E. A., & Ragos, O. (2006**). *Periodic orbits in the photogravitational restricted three-body problem. Celestial Mechanics and Dynamical Astronomy, 94, 135–150. https://doi.org/10.1007/s10569-005-6183-7*
15. **Roman, R., & Mioc, V. (2004).** *Photogravitational restricted three-body problem: Analytical investigations. Nonlinear Analysis: Real World Applications, 5(5), 735–747. https://doi.org/10.1016/j.nonrwa.2003.09.002*
16. **Zimovshchikov, D., & Tkhai, V. (2004).** *On the stability of Lagrangian points in the restricted three-body problem. Astronomy Reports, 48(9), 705–713. https://doi.org/10.1134/1.1796230*
17. **Haque, M., & Ishwar, B. (1995).** *Photogravitational restricted three-body problem with radiation pressure. Astrophysics and Space Science, 229, 1–10. https://doi.org/10.1007/BF00643519*
18. **Markeev, A. P., & Sokolski, V. (1977).** *Stability of motion in the restricted three-body problem. Celestial Mechanics, 15, 267–282. https://doi.org/10.1007/BF01228413*
19. **Breakwell, J. V., & Pringle, R. (1966).** *Stability of the triangular points in the restricted three-body problem. Astronomical Journal, 71, 196–199.*
20. **Whittaker, E. T. (1965).** *A treatise on the analytical dynamics of particles and rigid bodies. Cambridge University Press*

 DOI : 10.35291/2454-9150.2026.0037