Special Types of Domination Matrix in Graphs

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Abstract: **Let G (V, E) be a connected graph. A set S of vertices in a graph G = (V, E) is called a Dominating set of G if every vertex in V – S is adjacent to some vertex in S. The theory of Domination has been the nucleus of Research Activity in Graph Theory in recent times. This is largely due to a variety of new parameters that can be developed from the basic definition of domination. Matrix is a concise graph representation, but alone it can be insufficient. That is, without the proper ordering of matrix rows and columns, the underlying graph structure is not necessarily apparent. This paper aims at the study of new parameter of Domination matrices for connected graph G.**

Keywords — Domination Matrix, Independent Domination Matrix, Independent Domination Matrix, Clique Domination Matrix, Total Domination matrix and Global Domination Matrix.

I. Introduction

Let G = (V, E) be a simple graph of order p. For any v ∈ V(G), the **neighborhood** $N_G(v)$ (or simply N(v)) of v is the set of all vertices adjacent to v in G. A non – empty set S ⊆ V(G) of a graph G is a **Dominating Set**, if every vertex in V(G) – S is adjacent to atleast one vertex in S. A dominating set S ⊆ V is an **Independent Dominating set** if the induced subgraph <S> has no edges. The **Independent Domination Number** i(G) of a graph G is the minimum cardinality of an independent dominating set. A dominating set S ⊆ V is a **Total Dominating set** if the induced subgraph <S> has no isolated vertices. The total domination number $\gamma_t(G)$ of a graph G is the minimum cardinality of a total dominating set. This concept of total domination was introduced by Cockayne, Dawes and Hedetniemi in [1]. A clique dominating set was introduced by Cozzens and Kelleher [2]. A dominating set S ⊆ V is a **Clique Dominating set** if the induced subgraph <S> is a complete graph. The clique domination number $\gamma_{cl}(G)$ of a graph G is the minimum cardinality of a clique dominating set. A dominating set S of a graph G is a **Global Dominating Set** if S is also a dominating set of $\overline{G}$. The global domination number $\gamma_g(G)$ is the minimum cardinality of a global dominating set of G. This concept was introduced by Sampathkumar [6]. Many real-world Engineering Systems are too complex to be defined in precise terms, and, in many matrix equations, some or all of the system parameters are vague or imprecise. Rather than literally drawing graphs, it can be visualized the corresponding graph called Adjacency Graph. The Adjacency matrix represents each graph edge with a single matrix element, as opposed to a draw line. Graph vertices, rather than being drawn explicitly, are implicitly represented as matrix rows and columns, not only for very large graphs, but also for smaller ones. This paper aims at the study of the above defined parameters of Domination matrices for connected graph G.

II. PRELIMINARIES

Definition 2.1. **Matrix** is a rectangular array of numbers, symbols or expression arranged in rows and columns and usually it is denoted by capital letters.

Definition 2.2. Given a matrix A. The **Submatrix** is a matrix obtained from the matrix A by deleting some of the row(s) and/or column(s) of A.

Definition 2.3. A **(0, 1) – matrix** is an integer matrix in which each entry in the matrix is either '0' or '1'. It is also called Relation matrix.

Illustration 2.4. Consider the graph G as shown in Figure 2.5.

G ≅

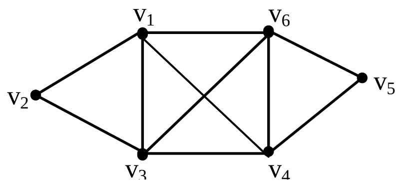

Figure 2.5.

Definition 2.6. Let G be a graph of order p then the **Adjacency matrix** A(G) is a p x p matrix with the row labeling $V_1, V_2, \ldots, V_p$ and the column labeling $V_1, V_2, \ldots, V_p$ corresponding to the vertices $v_1, v_2, \ldots, v_p$ is given by

$$a_{ij} = \begin{cases} 1; \text{if there is an edge between the vertices } v_i \text{ and } v_j \\ -; \text{if } i = j \\ 0; \text{otherwise} \end{cases}.$$

a_{ij} denote the entries of the adjacency matrix in the i^{th} row and j^{th} column for all $v_i, v_j \in V(G)$.

Example 2.7. The Adjacency matrix of the graph G in Figure 2.5. is given below

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$$A(G) = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{array} \begin{array}{c} \begin{matrix} \mathbf{V}_1 & \mathbf{V}_2 & \mathbf{V}_3 & \mathbf{V}_4 & \mathbf{V}_5 & \mathbf{V}_6 \end{matrix} \\ \begin{pmatrix} - & 1 & 1 & 1 & 0 & 1 \\ 1 & - & 1 & 0 & 0 & 0 \\ 1 & 1 & - & 1 & 0 & 1 \\ 1 & 0 & 1 & - & 1 & 1 \\ 0 & 0 & 0 & 1 & - & 1 \\ 1 & 0 & 1 & 1 & 1 & - \end{pmatrix} \end{array}$$

Remark 2.8.

i. Adjacency matrix of a graph G is a symmetric matrix since the graph is an undirected graph.

ii. Sum of the entries in a row(or a column) of the Adjacency matrix is equal to the degree of the corresponding vertex.

Definition 2.9. Let G be a connected graph and V(G) = {v_1, v_2, …, v_p} be the vertex set of the graph G. Let A(G) be the adjacency matrix of the graph G. There exist a row with label V_i in A(G) for all i = 1, 2, …, p. Then the **Row operation** on the Adjacency matrix is defined as follows

Let V_i and V_k be the two rows in the Adjacency matrix A(G) corresponding to the vertices v_i and v_k of the graph G. Then

$$V_i \oplus V_k = a_{ij} \oplus a_{kj} = \begin{cases} 0; & \text{if } a_{ij} = a_{kj} = 0 \\ -; & \text{if } i = j \text{ (or) } k = j \\ 1; & \text{otherwise} \end{cases} \quad \text{for all}$$

j = 1, 2, …, p.

Example 2.10. For the Adjacency matrix A(G) in Example 2.7. row operations are illustrated as follows

(i). If the Row operation $V_5 \oplus V_6$ is performed then the corresponding entries are

1 0 1 1 – –

(ii). If the Row operation $V_2 \oplus V_5$ is performed then the corresponding entries are

1 −1 1 −1

(iii). If the Row operation $V_1 \oplus V_6$ is performed then the corresponding entries are

– 1 1 1 1 –

Remark 2.11. After performing the row operation the entry at the position of the column will be '–' corresponding to the row labels included in the row operations. Hence after performing the row operations the number of entries with '–' are equal to the number of vertices included in the row operation.

Theorem 2.12[5]. The vertices included in the finite number of row operations of an Adjacency matrix will denote a dominating set if and only if there is no zero entry after performing the row operations.

Definition 2.13. Let G be a connected graph and S be a dominating set of G. **Domination matrix** is a matrix with columns labels are the vertices in the set S and the row labels are the vertices in the complement of the set S. **Domination matrix** is a submatrix of the adjacency matrix with the given column and row labels. The domination matrix is denoted by DM(G).

The domination matrix corresponding to the minimum dominating set will form a **Minimum Domination matrix.** It is denoted by $DM_\gamma(G)$.

Remark 2.14. All the entries in the domination matrix are either '0' or '1', there is no '-' entry since the labels in the rows and columns are different.

Example 2.15. In Example 2.10. the row operation (i) will not form a Dominating Set since there is a '0' in the second position.

In Example 2.10. the row operation (ii). form the Dominating Set and the corresponding Domination Matrix is given below.

$$DM_1(G) = \begin{array}{c} \\ \mathbf{V}_1 \\ \mathbf{V}_3 \\ \mathbf{V}_4 \\ \mathbf{V}_6 \end{array} \begin{array}{c} \begin{matrix} \mathbf{V}_2 & \mathbf{V}_5 \end{matrix} \\ \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \end{array}$$

In Example 2.10. the row operation (iii). form the Dominating set and the corresponding Domination Matrix is given below.

$$DM_2(G) = \begin{array}{c} \\ \mathbf{V}_2 \\ \mathbf{V}_3 \\ \mathbf{V}_4 \\ \mathbf{V}_5 \end{array} \begin{array}{c} \begin{matrix} \mathbf{V}_1 & \mathbf{V}_6 \end{matrix} \\ \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \end{array}$$

The dominating matrix $DM_1(G)$ and $DM_2(G)$ contains the minimum number of vertices. Therefore $DM_1(G)$ and $DM_2(G)$ are minimum dominating matrices.

Hence $DM_\gamma(G) = DM_1(G) = DM_2(G)$.

III. MAIN RESULTS

Definition 3.1: Let G be a connected graph, S be a dominating set of G and DM(G) be the **Domination matrix.** The Induced Domination Matrix is the submatrix of the adjacency matrix with the entries in the Dominating set S. It is denoted by <DM(G)>.

Example 3.2: In Example 2.10, the induced domination matrix $\langle DM_1(G)\rangle$ is given by

$$\langle DM_1(G)\rangle = \begin{array}{c} \\ \mathbf{V}_2 \\ \mathbf{V}_5 \end{array} \begin{array}{c} \begin{matrix} \mathbf{V}_2 & \mathbf{V}_5 \end{matrix} \\ \begin{pmatrix} - & 0 \\ 0 & - \end{pmatrix} \end{array}$$

In Example 2.10, the induced domination matrix $\langle DM_2(G)\rangle$ is given by

$$\langle DM_2(G)\rangle = \begin{array}{c} \\ \mathbf{V}_1 \\ \mathbf{V}_6 \end{array} \begin{array}{c} \begin{matrix} \mathbf{V}_1 & \mathbf{V}_6 \end{matrix} \\ \begin{pmatrix} - & 1 \\ 1 & - \end{pmatrix} \end{array}$$

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Theorem 3.3: A domination matrix of a graph G is an **Independent domination matrix** if and only if in the induced domination matrix all the entries are either '0' or '-' in all the rows.

Proof. Let G be a connected graph and S be an independent dominating set of G and <DMG> be the domination matrix of the set S. If in <DM(G)> all the entries are either '0' or '-' in all the rows then it will represent that the induced subgraph <S> has no edges. By the definition, it follows that <DM(G)> will be an independent domination matrix.

Conversely, assume that <DM(G)> be an independent domination matrix of the graph G. On the contrary, if there is atleast a '1' in any one row then it will indicate that there is an edge in the induced dominating set S, which is a contradiction. Hence the theorem follows.

Remark 3.4. The independent domination matrix is denoted by IDM(G). The independent domination matrix which includes minimum number of entries in the column will form a **Minimum independent Domination matrix.** It is denoted by $IDM_\gamma(G)$.

In Example 3.1. the induced domination matrix <$DM_1(G)$> is an Independent Domination Matrix, since all the entries are either '0' or '-'. Also $IDM_\gamma(G)$ = <$DM_1(G)$>.

Theorem 3.5: A domination matrix of a graph G is a clique domination matrix if and only if in the induced domination matrix all the entries are either '1' or '-' in all the rows.

Proof. Let G be a connected graph and S be a clique dominating set of G. If in the induced domination matrix <DM(G)> of the set S, all the entries are either '1' or '-' in all the rows then it will represent that the induced subgraph <S> is a complete graph. By the definition, it follows that <DM(G)> will be a clique domination matrix.

Conversely, assume that <DM(G)> be a clique domination matrix of the graph G. On the contrary, if there is atleast a '0' in any one row then it will indicate that there is not an edge in the induced dominating set S, which is a contradiction. Hence the theorem follows.

Remark 3.6. The clique domination matrix is denoted by CDM(G). The clique domination matrix which includes minimum number of entries in the column will form a **Minimum Clique Domination matrix.** It is denoted by $CDM_\gamma(G)$.

In Example 3.1. the induced domination matrix <$DM_2(G)$> is a Clique Domination Matrix, since all the entries are either '1' or '-'. Also $CDM_\gamma(G)$ = <$DM_2(G)$>.

Theorem 3.7: A domination matrix of a graph G is a Total domination matrix if and only if in the induced domination matrix there is atleast a '1' in all the rows.

Proof. Let G be a connected graph and S be a Total dominating set of G. If in the induced domination matrix <DM(G)> of the set S, there is atleast a '1' in all the rows then it will represent that the induced subgraph <S> has no isolated vertex. By the definition, it follows that <DM(G)> will be a total domination matrix.

Conversely, assume that <DM(G)> be a total domination matrix of the graph G. On the contrary, if there is atleast one row with all the entries to be '0', then it will indicate that there is an isolated vertex in the induced dominating set S, which is a contradiction. Hence the theorem follows.

Example 3.8: Consider the Graph G_1,

$G_1 \cong$

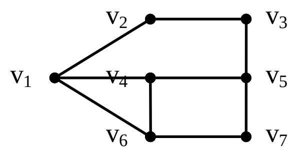

Figure 3.9.

Adjacency matrix of the graph G_1 is

$A(G_1)$ =

	V_1	V_2	V_3	V_4	V_5	V_6	V_7
V_1	–	1	0	1	0	1	0
V_2	1	–	1	0	0	0	0
V_3	0	1	–	0	1	0	0
V_4	1	0	0	–	1	1	0
V_5	0	0	1	1	–	0	1
V_6	1	0	0	1	0	–	1
V_7	0	0	0	0	1	1	–

For the Adjacency matrix $A(G_1)$ row operations are illustrated as follows

(i). If the Row operation $V_2 \oplus V_4 \oplus V_6$ is performed then the corresponding entries are

$$1 \; -1 \; -1 \; -1$$

(ii). If the Row operation $V_2 \oplus V_3 \oplus V_5 \oplus V_7$ is performed then the corresponding entries are

$$1 \; - \; -1 \; -1 \; -$$

The Domination Matrix corresponding to the row operations (i) and (ii) for the graph G_1 are

$DM_1(G_1)$ =

	V_2	V_4	V_6
V_1	1	1	1
V_3	1	0	0
V_5	0	1	0
V_7	0	0	1

$DM_2(G_1)$ =

	V_2	V_3	V_5	V_7
V_1	1	0	0	0
V_4	0	0	1	0
V_6	0	0	0	1

The induced domination matrix <$DM_1(G_1)$> and

<$DM_2(G_1)$> are <$DM_1(G_1)$> =

	V_2	V_4	V_6
V_2	–	0	0
V_4	0	–	1
V_6	0	1	–

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$$\langle DM_2(G_1)\rangle = \begin{array}{c|cccc} & V_2 & V_3 & V_5 & V_7 \\ \hline V_2 & - & 1 & 0 & 0 \\ V_3 & 1 & - & 1 & 0 \\ V_5 & 0 & 1 & - & 1 \\ V_7 & 0 & 0 & 1 & - \end{array}$$

The induced domination matrix $\langle DM_1(G_1)\rangle$ is not a total dominating matrix, since all the entries in the first row is zero. That is, the vertex v_2 is an isolated vertex.

The induced domination matrix $\langle DM_2(G_1)\rangle$ is a total dominating matrix, since there is no row with all the entries to be zero. That is, there is no isolated vertex in $DM_2(G_1)$.

Remark 3.10. The clique domination matrix is denoted by CDM(G). The clique domination matrix which includes minimum number of entries in the column will form a **Minimum Clique Domination matrix.** It is denoted by $CDM_\gamma(G)$.

In Example 3.9. the induced domination matrix $\langle DM_2(G)\rangle$ is a Clique Domination Matrix, since all the entries are either '1' or '-'. Also $CDM_\gamma(G) = \langle DM_2(G)\rangle$.

Definition 3.11: Adjacency matrix of the complement of the graph G that is, $\overline{G}$ is obtained by replacing the entries '0' by '1' and '1' by '0'.

Example 3.12:

$\overline{G} \cong$

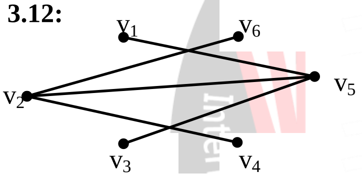

Figure 3.13.

The Adjacency matrix of the graph $\overline{G}$ is

$$A(\overline{G}) = \begin{array}{c|cccccc} & V_1 & V_2 & V_3 & V_4 & V_5 & V_6 \\ \hline V_1 & - & 0 & 0 & 0 & 1 & 0 \\ V_2 & 0 & - & 0 & 1 & 1 & 1 \\ V_3 & 0 & 0 & - & 0 & 1 & 0 \\ V_4 & 0 & 1 & 0 & - & 0 & 0 \\ V_5 & 1 & 1 & 1 & 0 & - & 0 \\ V_6 & 0 & 1 & 0 & 0 & 0 & - \end{array}$$

The graph $\overline{G}$ is connected since all the rows and columns contains atleast a '1'.

Theorem 3.13: A domination matrix of a graph G is a Global domination matrix if and only if in the domination matrix there is atleast a '1' as well as a '0' in all the rows.

Proof. Let G be a connected graph and S be a Global dominating set of G. Let $\overline{G}$ be the complement of the graph G. Let DM(G) be the domination matrix corresponding to the set of the graph G. If in DM(G) there is atleast a '1' in all the rows then it will represent that S is a Dominating set of the graph G. If in addition, there is atleast a '0' in all the rows then it will represent that DM(G) is a Dominating set of the complement graph $\overline{G}$. By the definition, it follows that DM(G) will be a global domination matrix.

Conversely, assume that DM(G) be a global domination matrix of the graph G. On the contrary, if there is atleast one row with all the entries to be '0', then it will indicate that S is not a dominating set of $\overline{G}$, which is a contradiction. If in addition, there is atleast one row with all the entries to be '1', then it will indicate that S is not a dominating set of G, which is again a contradiction. Hence the theorem follows.

Illustration 3.14: For the Adjacency matrix $A(\overline{G})$ in Example 3.12., the row operations are as follows

(i). If the Row operation $V_2 \oplus V_5$ is performed then the corresponding entries are

$$1 \quad -1 \quad 1 \quad -1$$

The Dominating Set and the corresponding Domination Matrix of the graph $\overline{G}$ is given below.

$$DM_1(\overline{G}) = \begin{array}{c|cc} & V_2 & V_5 \\ \hline V_1 & 0 & 1 \\ V_3 & 0 & 1 \\ V_4 & 1 & 0 \\ V_6 & 1 & 0 \end{array}$$

Remark 3.15. The global domination matrix is denoted by GDM(G). The global domination matrix which includes minimum number of entries in the column will form a **Minimum Global Domination matrix.** It is denoted by $GDM_\gamma(G)$.

In Illustration 3.14. the domination matrix $DM_1(\overline{G})$ is a Global Domination Matrix. Also, $GDM_\gamma(G) = DM_1(\overline{G})$.

Note 3.16. If a domination matrix DM(G) is a global domination matrix then its complement $DM(\overline{G})$ will also be a global domination matrix since, if there is atleast a '1' and '0' in all the rows of G will indicate that there is atleast a '0' and a '1' in all the rows of $\overline{G}$.

Therefore the domination matrix $DM_1(G)$ in Example 2.15, is a global domination matrix.

IV. CONCLUSION

Thus the different types of Domination matrices namely Independent Domination Matrix, Clique Domination Matrix, Total Domination Matrix and Global Domination Matrix are established. Further determining the Inverse Domination Matrix and Degree Equitable Domination Matrix are the future work.

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REFERENCES

[1] Cockayne C. J., Dawes R. M. and Hedetniemi S.T., Total Domination in Graphs, *Networks*, Vol. 10 (1980), Pp. 211 – 219.

[2] Cozzens M.B. and Kelleher L. L., Dominating Cliques in Graphs, *Discrete Math.*, Vol. 86 (1990), Pp. 101 – 116.

[3] Harary F, Graph Theory, Addison – Wesley, Reading Mass, 1969.

[4] Haynes T. W, Hedetniemi S. T, Slater P. J, Fundamentals of Domination in Graphs, Marcel Dekker Inc., 1998.

[5] Meenal N., Co-isolated Locating Domination Matrix in Graphs, *International Review of Pure and Applied Mathematics*, Accepted for Publication, July – December 2018.

[6] Sampathkumar E., The Global Domination Number of a Graph, *Journal of Math. Phys. Science*., Vol. 23 (1989), Pp. 377 – 385.

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